An Algebraic Theory of the Commons

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November 16, 2009
Number of herdsmen: $h$

The average number of cattle (at the limit of the grazing): $n$

The number of the herd at this limit: $N = n \times h$

Percentage loss in value: $a$

number of heads of cattle beyond grazing capacity $x$

The total value of herd: $(N + x)(1 - ax)$
GAME

- 2 players:
  1. individual herdsman
  2. all the other herdsmen

- Two choices:
  1. Adding one head of cattle
  2. not adding one head of cattle

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Add</th>
<th>Do not add</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adds</td>
<td>(X, Y)</td>
<td>(Z, W)</td>
</tr>
<tr>
<td>Does not add</td>
<td>(T, U)</td>
<td>0, 0</td>
</tr>
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Normal Form Game

First CASE: 
\[ X = (1 - a(N + h)), \ Y = ((h - 1)(1 - a(N + h))) \]

\[ U_1(\text{Adds}/\text{Add}) = (n + 1)(1 - ah) - n \]
\[ U_2(\text{Adds}/\text{Add}) = (h - 1)(n + 1)(1 - ah) - n(h - 1) \]

Second CASE: \[ Z = (1 - a(n + 1)), \ W = (-a(N - n)) \]

\[ U_1(\text{Adds}/\text{Do not add}) = (n + 1)(1 - a) - n \]
\[ U_2(\text{Donotadd}/\text{Add}) = (h - 1)(n)(1 - a) - n(h - 1) \]
Normal Form Game

- Third CASE:
  \[ T = (-a(N - n)), U = ((h - 1)(1 - a(h - 1)(n + 1))) \]

1. \( U_1\) (Does not add/Add) = \( (n)(1 - a(h - 1)) - n \)
2. \( U_2\) (Adds/Does not add) = \( (h - 1)(n + 1)(1 - a(h - 1)) - n(h - 1) \)

- Second CASE: [0, 0]

1. \( U_1\) (Does not add/Do not add) = 0
2. \( U_2\) (Do not add/Does not add) = 0

**Player 2**

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