'maximum sustainable yield'. But for a stand of trees all planted at one point in time, the concept of a sustainable yield of timber is not meaningful (except for specialised activities such as coppicing). While one can conceive of harvesting mature fish while leaving younger fish to grow to maturity, this cannot happen on a continuous basis in a single-aged forest stand. However, when there are many stands of trees of different ages, it is meaningful to talk about sustainable yields. This is something we shall discuss later.

18.3 Commercial plantation forestry

There is a huge literature dealing with efficient timber extraction. We attempt to do no more than present a flavour of some basic results, and refer the reader to specialist sources of further reading at the end of the chapter. An economist derives the criterion for an efficient forest management and felling programme by trying to answer the following question:

What harvest programme is required in order that the present value of the profits from the stand of timber is maximised?

The particular aspect of this question that has most preoccupied forestry economists is the appropriate time after planting at which the forest should be felled. As always in economic analysis, the answer one gets to any question depends on what model is being used. We begin with one of the most simple forest models, the single-rotation commercial forest model. Despite its lack of realism, this model offers useful insights into the economics of timber harvesting. However, as we shall see later in the chapter, that which is privately optimal may not be socially efficient. In particular, where private costs and benefits fail to match their social counterparts, a wedge may be driven between privately and socially efficient behaviour. For the moment, we put these considerations to one side.

18.3.1 A single-rotation forest model

Suppose there is a stand of timber of uniform type and age. All trees in the stand were planted at the same time, and are to be cut at one point in time. Once felled, the forest will not be replanted. So only one cycle or rotation – plant, grow, cut – is envisaged. For simplicity, we also assume that

- the land has no alternative uses so its opportunity cost is zero;
- planting costs \((k)\), marginal harvesting costs \((c)\) and the gross price of felled timber \((P)\) are constant in real terms over time;
- the forest generates value only through the timber it produces, and its existence (or felling) has no external effects.

Looked at from the point of view of the forest owner (which, for simplicity, we take to be the same as the landowner), what is the optimum time at which to fell the trees? The answer is obtained by choosing the age at which the present value of profits from the stand of timber is maximised. Profits from felling the stand at a particular age of trees are given by the value of felled timber less the planting and harvesting costs. Notice that because we are assuming the land has no other uses, the opportunity cost of the land is zero and so does not enter this calculation. If the forest is clear-cut at age \(T\), then the present value of profit is

\[
(P - c)S_0e^{-iT} - k = pS_0e^{-iT} - k
\]

(18.1)

where \(S_0\) denotes the volume of timber available for harvest at time \(T\), \(p\) (in lower-case, note) is the net price of the harvested timber, and \(i\) is the private consumption discount rate (which we suppose is equal to the opportunity cost of capital to the forestry firm).

The present value of profits is maximised at that value of \(T\) which gives the highest value for \(pS_0e^{-iT} - k\). To maximise this quantity, we differentiate equation 18.1 with respect to \(T\), using the product rule, set the derivative equal to zero and solve for \(T\):

\[\text{Note from the first of these steps that } k \text{ does not enter the first derivative, and so immediately we find that in a single rotation model, planting costs have no effect on the efficient rotation length provided that } k \text{ is not so large as to make the maximised present value negative.}\]
\[
\frac{d}{dT}(pS_re^{-\tau} - k) = \frac{d}{dT}(pS_re^{-\tau})
\]
\[
= p e^{-\tau} \frac{dS}{dT} + pS_r \frac{d}{dT} e^{-\tau}
\]
which, setting equal to zero, implies that
\[
p e^{-\tau} \frac{dS}{dT} = ipS_r e^{-\tau} = 0
\]
and so
\[
\frac{dS}{dT} = ipS_r
\]

or
\[
i = \frac{\frac{dS}{dT}}{pS_r} \tag{18.2}
\]

Equation 18.2 states that the present value of profits is maximised when the rate of growth of the (undiscounted) net value of the resource stock is equal to the private discount rate. Note that with the timber price and harvesting cost constant, this can also be expressed as an equality between the proportionate rate of growth of the volume of timber and the discount rate. That is,

\[
\frac{dS}{dT} = \frac{dS}{S_r} t
\]

We can calculate the optimal, present-value-maximising age of the stand for the illustrative data in Table 18.3. These calculations, together with the construction of the associated graphs are reproduced in the Excel workbook Chapter18.xls which can be downloaded from the Additional Materials web page. In these calculations, we assume that the market price per cubic foot of felled timber is £10, total planting costs are £5000, incurred immediately the stand is established, and harvesting costs are £2 per cubic foot, incurred at whatever time the forest is felled. The columns labelled R1, C1 and NB1 list the present values of revenues and costs and profits (labelled Net benefit in the table) for a discount rate of zero. Note that when \(i = 0\), present values are identical to undiscounted values. The level of the present value of profits (NB1) over time is shown in Figure 18.2. Net benefits are maximised at 115 years, the point at which the biological growth of the stand (dS/dt) becomes zero. With no discounting and fixed timber prices, the profile of net value growth of the timber is identical to the profile of net volume growth of the timber, as can be seen by comparing Figures 18.1(a) and 18.2.

![Figure 18.2 Present values of net benefits at \(i = 0.00\) (NB1) and \(i = 0.03\) (NB2)](image_url)
It is also useful to look at this problem in another way. The interest rate for a forest owner is the opportunity cost of the capital tied up in the growing timber stand. When the interest rate is zero, that opportunity cost is zero. It will, therefore, be in the interests of the owner to not harvest the stand as long as the volume (and value) growth is positive, which it is up to an age of 135 years. Indeed, inspection of equation 18.2 confirms this; given that \( S \) is positive, when \( i = 0 \) \( \frac{ds}{dt} \) must be zero to satisfy the first-order maximising condition.

Now consider the case where the discount rate is 3%. The columns labelled R2, C2 and NB2 in Table 18.3 refer to the present values of revenues, costs and profits when the interest rate is 3%. The present value of profits at a discount rate of 3% is also plotted in Figure 18.2, under the legend NB2. With a 3% discount rate, the present value of the forest is maximised at a stand age of 50 years.

Expressed in a way that conforms to equation 8.2, the growth of undiscounted profits,

\[
\frac{dS}{dt} \frac{p}{pS_T}
\]

equals \( i \) (at 3%) in year 50, having been larger than 3% before year 50 and less than 3% thereafter. This is shown by the ‘i = 3%’ line which has an identical slope to that of the NB1 curve at \( t = 50 \) in Figure 18.2. At that point, the growth rate of undiscounted timber value equals the interest rate. A wealth-maximising owner should harvest the timber when the stand is of age 50 years – up to that point, the return from the forest is above the interest rate, and beyond that point the return to the forest is less than the interest rate.

The single-rotation model we have used shows that the optimal time for felling will depend upon the discount rate used. It can be seen from our calculations that this effect can be huge. A rise in the discount rate from zero to 3% not only dramatically lowers the profitability of the forest but also significantly changes the shape of the present-value profile, reducing the age at which the forest should be felled (in our illustrative example) from 135 to 50 years.

More generally, it is clear from our previous arguments that as the interest rate rises the age at which the stand is felled will have to be lowered in order to bring about equality between the rate of change of undiscounted net benefits and the discount rate. In Figure 18.3, we illustrate how the optimal felling age varies with the interest rate for our illustrative data. While the exact relationship shown is only valid under the assumptions used here, it does suggest that
small changes in interest rates might dramatically alter privately optimal harvesting programmes.

18.3.2 Infinite-rotation forestry models

The forestry model we investigated in the previous section is unsatisfactory in a number of ways. In particular, it is hard to see how it would be meaningful to have only a single rotation under the assumption that there is no alternative use of the land. If price and cost conditions warranted one cycle then surely, after felling the stand, a rational owner would consider further planting cycles if the land had no other uses? So the next step is to move to a model in which more than one cycle or rotation occurs. The conventional practice in forestry economics is to analyse harvesting behaviour in an infinite time horizon model (in which there will be an indefinite quantity of rotations). A central question investigated here is what will be the optimal length of each rotation (that is, the time between one planting and the next).

When the harvesting of one stand of timber is to be followed by the establishment of another, an additional element enters into the calculations. In choosing an optimal rotation period, a decision to defer harvesting incurs an additional cost over that in the previous model. We have already taken account of the fact that a delay in harvesting has an opportunity cost in the form of interest forgone on the (delayed) revenues from harvesting. But a second kind of opportunity cost now enters into the calculus. This arises from the delay in establishing the next and all subsequent planting cycles. Timber that would have been growing in subsequent cycles will be planted later. So an optimal harvesting and replanting programme must equate the benefits of deferring harvesting – the rate of growth of the undiscounted net benefit of the present timber stand – with the costs of deferring that planting – the interest that could have been earned from timber revenues and the return lost from the delay in establishing subsequent plantings.

Our first task is to construct the present-value-of-profits function to be maximised for the infinite-rotation model. We continue to make several simplifying assumptions that were used in the single-rotation model: namely, the total planting cost, k, the gross price of timber, P, and the harvesting cost of a unit of timber, c, are constant through time. Given this, the net price of timber \( p = P - c \) will also be constant.

Turning now to the rotations, we assume that the first rotation begins with the planting of a forest on bare land at time \( t_0 \). Next, we define an infinite sequence of points in time that are ends of the successive rotations, \( t_1, t_2, t_3, \ldots \). At each of these times, the forest will be clear-felled and then immediately replanted for the next cycle. The net present value of profit from the first rotation is

\[
pS_{t_0-t_0}e^{-r(t_0-t_0)} - k
\]

that is, the volume of timber growth between the start and end of the cycle multiplied by the discounted net price of a unit of timber, less the forest planting cost. Notice that because the planting cost is incurred at the start of the rotation, no discounting is required to bring it into present-value terms. But as the timber is felled at the end of the rotation \( t_1 \), the timber revenue has to be discounted back to its present \( t_0 \) value equivalent.

The net present value of profits over this infinite sequence is given by

\[
\Pi = [pS_{t_0-t_0}e^{-r(t_0-t_0)} - k] \\
+ e^{-r(t_0-t_1)}[pS_{t_0-t_1}e^{-r(t_0-t_1)} - k] \\
+ e^{-r(t_0-t_2)}[pS_{t_0-t_2}e^{-r(t_0-t_2)} - k] \\
+ e^{-r(t_0-t_3)}[pS_{t_0-t_3}e^{-r(t_0-t_3)} - k] \\
+ \ldots
\]

(18.3)

Reading this, we see that the present value of profits from the infinite sequence of rotations is equal to the sum of the present values of the profit from each of the individual rotations.

Provided conditions remain constant through time, the optimal length of any rotation will be the same as the optimal length of any other. Call the interval of time in this optimal rotation \( T \). Then we can rewrite the present-value function as

\[
\Pi = [pS_T e^{-rT} - k] \\
+ e^{-rT}[pS_T e^{-rT} - k] \\
+ e^{-2rT}[pS_T e^{-rT} - k] \\
+ e^{-3rT}[pS_T e^{-rT} - k] \\
+ \ldots
\]

(18.4)
Next, factorise out \(e^{-iT}\) from the second term on the right-hand side of equation 18.4 onwards to give

\[
\Pi = [pS_T e^{-iT} - k] + e^{-iT} \left[ \frac{(pS_T e^{-iT} - k)}{1 - e^{-iT}} \right] + \ldots \]

(18.5)

Now look at the term in brackets on the right-hand side of equation 18.5. This is identical to \(\Pi\) in equation 18.4. Therefore, we can rewrite equation 18.5 as

\[
\Pi = [pS_T e^{-iT} - k] + e^{-iT} \Pi
\]

(18.6)

which on solving for \(\Pi\) gives

\[
\Pi = \frac{pS_T e^{-iT} - k}{1 - e^{-iT}}
\]

(18.7)

Equation 18.7 gives the present value of profits for any rotation length, \(T\), given values of \(p\), \(k\), \(i\) and the timber growth function \(S = S(t)\). The wealth-maximising forest owner selects that value of \(T\) which maximises the present value of profits. For the illustrative data in Table 18.3, we have used a spreadsheet program to numerically calculate the present-value-maximising rotation intervals for different values of the discount rate. (The spreadsheet is available in Additional Materials, Chapter 18, as Chapter18.xls, Sheet 2.) Present values were obtained by substituting the assumed values of \(p\), \(k\) and \(i\) into equation 18.7, and using the spreadsheet to calculate the value of \(\Pi\) for each possible rotation length, using Clawson’s timber growth equation. The results of this exercise are presented in Table 18.4 (along with the optimal rotation lengths for a single rotation forest, for comparison). Discount rates of 6% or higher result in negative present values at any rotation, and the asterisked rotation periods shown are those which minimise present-value losses; commercial forestry would be abandoned at those rates. With our illustrative data, at any discount rate which yields a positive net present value for the forest the optimal rotation interval in an

<table>
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<tr>
<th>Table 18.4 Optimal rotation intervals for various discount rates</th>
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Notes to table:
1. Data simulated by Excel, using workbook Chapter18.xls
2. * For both single- and infinite-rotation models, at interest rates of 6% and above (for the price, cost and growth data used here) the PV is negative even at optimal \(T\), so the land would not be used for commercial forestry. The value of \(T\) shown in these cases is that which minimises the PV loss.
3. To simulate the solution for \(i = 0\), we used a value of \(i\) sufficiently close to (although not exactly equal to) zero so that the optimal rotation, in units of years, was unaffected by a further reduction in the value of \(i\).

infinite-rotation forest is lower than the age at which a forest would be felled in a single rotation model. For example, with a 3% discount rate, the optimal rotation interval in an infinite sequence of rotations is 40 years, substantially less than the 50-year harvest age in a single rotation. We will explain why this is so shortly.

It is also useful to think about the optimal rotation interval analytically, as this will enable us to obtain some important comparative statics results. Let us proceed as was done in the section on single-rotation forestry. The optimal value of \(T\) will be that which maximises the present value of the forest over an infinite sequence of planting cycles. To find the optimal value of \(T\), we obtain the first derivative of \(\Pi\) with respect to \(T\), set this derivative equal to zero, and solve the resulting equation for the optimal rotation length.

---

9 A more elegant method of obtaining equation 18.7 from 18.4 is as follows. Equation 18.4 may be rewritten as

\[
\Pi = \left( pS_T e^{-iT} - k \right) \left( 1 + e^{-iT} + (e^{-iT})^2 + \ldots \right)
\]

The final term in parentheses is the sum of an infinite geometric progression. Given the values that \(i\) and \(T\) may take, this is a convergent sum. Then, using the result for such a sum, that term can be written as \(1/(1 - e^{-iT})\), and so

\[
\Pi = \frac{pS_T e^{-iT} - k}{1 - e^{-iT}}
\]
The algebra here is simple but tedious, and so we have placed it in Appendix 18.1. Two forms of the resulting first-order condition are particularly useful, each being a version of the Faustmann rule (derived by the German capital theorist Martin Faustmann in 1849; see Faustmann (1968)). The first is given by

$$\frac{p}{pS_T} \frac{dS_T}{dT} = \frac{i}{1 - e^{-\alpha}} + \frac{i\Pi}{pS_T} \quad (18.8a)$$

and the second, after some rearrangement of 18.8a, is given by

$$p \frac{dS_T}{dT} = ipS_T + i\Pi \quad (18.8b)$$

Either version of equation 18.8 is an efficiency condition for present-value-maximising forestry, and implicitly determines the optimal rotation length for an infinite rotation model in which prices and costs are constant. Given knowledge of the function $S = S(t)$, and values of $p$, $i$ and $k$, one could deduce which value of $T$ satisfies equation 18.8 (assuming the solution is unique, which it usually will be). The term $\Pi$ in equation 18.8b is called the site value of the land – the capital value of the land on which the forest is located. This site value is equal to the maximised present value of an endless number of stands of timber that could be grown on that land.

The two versions of the Faustmann rule offer different advantages in helping us to make sense of optimal forest choices. Equation 18.8b gives some intuition for the choice of rotation period. The left-hand side is the increase in the net value of the timber left growing for an additional period. The right-hand side is the value of the opportunity cost of this choice, which consists of the interest forgone on the capital tied up in the growing timber (the first term on the right-hand side) and the interest forgone by not selling the land at its current site value (the second term on the right-hand side). An efficient choice equates the values of these marginal costs and benefits. More precisely, equation 18.8b is a form of Hotelling dynamic efficiency condition for the harvesting of timber. This is seen more clearly by rewriting the equation in the form:

$$p \frac{dS}{dT} = i + \frac{i\Pi}{pS_T} \quad (18.9)$$

Equation 18.9 states that, with an optimal rotation interval, the proportionate rate of return on the growing timber (the term on the left-hand side) is equal to the rate of interest that could be earned on the capital tied up in the growing timber (the first term on the right-hand side) plus the interest that could be earned on the capital tied up in the site value of the land ($i\Pi$) expressed as a proportion of the value of the growing timber ($pS_T$).

We can use the other version of the Faustmann rule – equation 18.8a – to illustrate graphically how the optimal rotation length is determined. This is shown in Figure 18.4. The curves labelled 0%, 1%, 2% and 3% plot the right-hand side of equation 18.8a for these rates of interest. The other, more steeply sloped, curve plots the left-hand side of the equation. At any given interest rate, the intersection of the functions gives the optimum $T$. The calculations required to generate Figure 18.4 are implemented in Sheet 3 of the Excel file Chapter18.xls, together with the chart itself.

The lines plotting the right-hand side of equation 18.8a are generated assuming particular values for $P$, $c$, $k$ and $i$, and also a particular natural growth function describing how timber volume $S$ changes over time. The reader is invited to copy this worksheet, and to study the way in which optimised $T$ varies as $p$ (that is, $P - c$), or $k$ changes, ceteris paribus.

### 18.3.2.1 Comparative static analysis

The results of the previous section have shown that in the infinite-rotation model the optimum rotation depends on:

- the biological growth process of the tree species in the relevant environmental conditions;
- the interest (or discount) rate ($i$);
- the cost of initial planting or replanting ($k$);
- the net price of the timber ($p$), and so its gross price ($P$) and marginal harvesting cost ($c$).

---

6 Unlike in the case of a single-rotation model, planting costs $k$ do enter the first derivative. So in an infinite-rotation model, planting costs do affect the efficient rotation length.
Comparative static analysis can be used to make qualitative predictions about how the optimal rotation changes as any of these factors vary. We do this algebraically using equation 18.8b. Derivations of the results are given in Appendix 18.2. Here we just state the results (for convenience, they are tabulated in Table 18.5) and provide some intuition for each of them.

Changes in the interest rate

The result that $dT/di < 0$ means that the interest rate and the optimal rotation period are negatively related. An increase (decrease) in $i$ causes a decrease (increase) in $T$. Why does this happen? Once planted, there are costs and benefits in leaving a stand unfelled for a little longer. The marginal benefit derives from the marginal revenue product of the additional timber growth. The marginal costs are of two kinds: first, the interest earnings forgone in having capital (the growing timber) tied up a little longer; and second, the interest earnings forgone from not clearing and then selling the bare land at its capital (site) value. If the interest rate increases, the terms of this trade-off change, because the opportunity costs of deferring felling become larger. Foresters respond to this by shortening their forest rotation period.

Changes in planting costs

The result that $dT/dk > 0$ means that a change in planting costs changes the optimal rotation in the same direction. A fall in $k$, for example, increases the site value of the land, $\Pi$. With planting costs lower, the profitability of all future rotations will rise, and so the opportunity costs of delaying replanting will rise. The next replanting should take place sooner. The optimal stand age at cutting will fall.

Changes in the net price of timber

The result that $dT/dp < 0$ means that the net price of timber ($p$) and the optimal rotation period are negatively related. Therefore, an increase in timber prices ($P$) will decrease the rotation period, and an increase in harvest costs will increase the rotation period. We leave you to deduce the intuition behind this for yourself.
yourself, in the light of what we have suggested for the two previous cases.

An Excel spreadsheet model (palc18.xls) can be used to explore these changes quantitatively, for an assumed growth process and particular values of the relevant economic parameters. We recommend that you work through that Excel file, and then experiment further with it. The workbook allows you to reproduce the numbers given in the textbook, to answer the Problems at the end of the chapter, and to see how the comparative static results work out quantitatively.

18.3.2.2 Comparing single and infinite rotations: how does a positive site value affect the length of a rotation?

To see the effect of land site values on the optimal rotation interval, compare equation 18.9 (the Hotelling rule taking into consideration positive site values) with equation 18.10, which is the Hotelling rule when site values are zero (and is obtained by setting $\Pi = 0$ in equation 18.9):

$$\frac{P \frac{dS}{dT}}{pS_r} = i$$  \hspace{1cm} (18.10)

In this case, an optimal rotation interval is one in which the rate of growth of the value of the growing timber is equal to the interest rate on capital alone.

But it is clear from inspection of equation 18.9 that for any given value of $i$, a positive site value will mean that $(dS/dt)/S$ will have to be larger than when the site value is zero if the equality is to be satisfied. This requires a shorter rotation length, in order that the rate of timber growth is larger at the time of felling. Intuitively, the opportunity cost of the land on which the timber is growing requires a compensating increase in the return being earned by the growing timber. With fixed timber prices, this return can only be achieved by harvesting at a point in time at which its biological growth is higher, which in turn requires that trees be felled at a younger age. Moreover, the larger is the site value, the shorter will be the optimal rotation.

It is this which explains why the optimal rotation intervals (for forests that are commercially viable) shown in Table 18.3 are shorter for infinite rotations than for a single rotation. In an infinite-rotation model, land is valuable (because the timber that can be grown on it in the future can yield profits), and the final term in equation 18.9 comes into play.

The reader should note that the way in which bare land is valued by the Faustmann rule – the present value of profits from an infinite sequence of optimal timber rotations – is not the only basis on which one might choose to arrive at land values. Another method would be to value the land at its true opportunity cost basis – that is, the value of the land in its most valuable use other than forestry. In many ways, this is a more satisfactory basis for valuation. This approach can give some insights into forestry location. In remote areas with few alternative land uses, low land prices may permit commercial forest growth even at high altitude where the intrinsic rate of growth of trees is low. In urban areas, by contrast, the high demand for land is likely to make site costs high. Timber production is only profitable if the rate of growth is sufficiently high to offset interest costs on tied-up land capital costs. There may be no species of tree that has a fast enough growth potential to cover such costs. In the same way, timber production may be squeezed out by agriculture where timber growth is slow relative to crop potential (especially where timber prices are low). All of this suggests that one is not likely to find commercial plantations of slow-growing hardwood near urban centres unless there are some additional values that should be brought into the calculus. It is to this matter that we now turn.

18.4 Multiple-use forestry

In addition to the timber values that we have been discussing so far, forests are capable of producing a wide variety of non-timber benefits. These include soil and water control, habitat support for a biologically diverse system of animal and plant populations, recreational and aesthetic amenities, wilderness existence values, and climate control. Where forests do provide one or more of these benefits to a significant extent, they are called multiple-use forests.