Full Rank Rational Demand Systems

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Abstract: We find the complete solution for the integrability problem of all full rank Gorman and Lewbel demand systems. All are special cases of a projective transformation group. We also find necessary and sufficient conditions for full rank and global regularity of a set of directly specified indirect utility functions that give rational demand systems. We derive a new class of flexible demand models to nest these two distinct approaches. The demands for this new class of models can have arbitrarily large full rank and be used to test for aggregation, global regularity, functional form, and flexibility of the demand system. Virtually all existing and well-known indirect utility functions are special cases of this new class of models, although none currently improve on the rank and flexibility properties of a Gorman or Lewbel system, and only one is globally regular.

Key Words: Aggregation, functional form, integrability, rank, rational demand systems

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1. Introduction

The two main approaches to economic modeling of smooth demand systems are to solve the integrability problem for a set of demand equations or to use Roy's Identity to generate demands. The first approach specifies an attractive set of demand equations $\boldsymbol{q}(\boldsymbol{p}, m)$ where $\boldsymbol{q}$ is the consumption vector, $\boldsymbol{p}$ is the associated price vector, and m is income.[1] The most common class of demand models of this type has the additive form,[2]

$$q_i = \sum_{k=1}^{K} \alpha_{ik}(\boldsymbol{p}) h_k(m), \; i = 1, ..., n, \tag{1}$$

where $\alpha_{ik} : \mathbb{R}^n_{++} \to \mathbb{R}, \;\; h_k : \mathbb{R}_{++} \to \mathbb{R}, \; \forall \; i = 1, \cdots, n, \; \forall \; k = 1, \cdots, K.$ [3] Define the $n \times K$ matrix of price functions $\boldsymbol{A}(\boldsymbol{p}) = \left[\alpha_{ik}(\boldsymbol{p})\right]$. The *rank* of (1) is the column rank of $\boldsymbol{A}(\boldsymbol{p})$, with $n \geq K$ (Gorman, 1981). *Full rank* systems are important because they are *parsimonious*. In parsimonious systems, for any given degree of flexibility in prices and income, the minimum number of parameters needs to be estimated. As a result, the main focus in the literature has been on full rank systems.

An important strength of (1) is that it aggregates from micro- to macro-level data; given the distribution function for income $F : \mathbb{R}_+ \to [0,1]$, we need the K moments,

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[1] We use the sobriquet *income* to denote total consumption expenditure, $m = e(\boldsymbol{p}, u) \equiv \min_{\boldsymbol{q} \in \mathbb{R}^n_+} \left\{ \boldsymbol{p}^\top \boldsymbol{q} : u(\boldsymbol{q}) \geq u \right\}$, where $u : \mathbb{R}^n_+ \to \mathbb{R}$ is the consumer's utility function.

[2] A small, but important, subset of the literature on this topic includes: Gorman (1953, 1961, 1965, 1981); Burt and Brewer (1971); Diewert (1971); Phlips (1971); Muellbauer (1975, 1976); Cicchetti, Fisher and Smith (1976); Howe, Pollak and Wales (1979); Deaton and Muellbauer (1980); Jorgenson, Lau and Stoker (1980, 1981, 1982); Russell (1983, 1996); Jorgenson and Slesnick (1984, 1987); Lewbel (1987a, 1988; 1989a, 1989b, 1990, 1991, 2003, 2004); Jorgenson (1990); Diewert and Wales (1987, 1988); Blundell (1988); Wales and Woodland (1988); Brown and Walker (1989); van Daal and Merkies (1989); Jerison (1993); Russell and Farris (1993, 1998); and Banks, Blundell, and Lewbel (1997). Consistent with this literature, we focus on smooth demand systems with interior solutions.

[3] The $n \times 1$ vector-valued price functions, $\{\boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_K\}$, and real-valued income functions, $\{h_1, \cdots, h_K\}$, are assumed throughout to be linearly independent across the K–dimensional constants to guarantee a unique representation (see Gorman (1981) or the Appendix of Russell and Farris (1998) by Robert Bryant). All functions also are assumed throughout to be smooth (i.e., $\alpha_{i,k}, h_k \in \mathcal{C}^\infty, \; \forall \; i = 1, \cdots, n, \; k = 1, \cdots, K$).

$$\bar{h}_k = \int_0^\infty h_k(m)dF(m),\ k = 1,...,K, \tag{2}$$

to estimate (1) with aggregate data. But a difficulty with (1) is the problem of integrability to well-behaved preferences (Hurwicz and Uzawa, 1971). For this class of models (hereafter a *Gorman system*), the demands must satisfy 0° homogeneity, adding up, and symmetry (and negative semi-definiteness) of the Slutsky equations,

$$\begin{aligned}\frac{\partial^2 e}{\partial p_i \partial p_j} &= \sum_{k=1}^K \frac{\partial \alpha_{ik}}{\partial p_j} h_k + \sum_{k=1}^K \alpha_{ik} h_k' \sum_{\ell=1}^K \alpha_{j\ell} h_\ell \\ &= \sum_{k=1}^K \frac{\partial \alpha_{jk}}{\partial p_i} h_k + \sum_{k=1}^K \alpha_{jk} h_k' \sum_{\ell=1}^K \alpha_{i\ell} h_\ell = \frac{\partial^2 e}{\partial p_j \partial p_i},\ \forall\ i \neq j,\end{aligned} \tag{3}$$

where $e : \mathbb{R}_+^n \times \mathbb{R} \to \mathbb{R}_+$ is the consumer's expenditure function. A great deal of emphasis has been on symmetry in previous studies.

The symmetry conditions (3) are identical to those discovered by Sophus Lie (1880; English translation with commentary in Hermann, 1975) in his seminal study of transformation groups. Russell (1982, 1996)[4] pointed out that a full rank Gorman system is a group transformation and that every full rank Gorman system is a special case of the quadratic form,

$$\frac{\partial y}{\partial \boldsymbol{p}} = \tilde{\boldsymbol{\alpha}}_1(\boldsymbol{p}) + \tilde{\boldsymbol{\alpha}}_2(\boldsymbol{p})y + \tilde{\boldsymbol{\alpha}}_3(\boldsymbol{p})y^2, \tag{4}$$

where $y = f(e(\boldsymbol{p},u))$ is a smooth and strictly monotonic function of total expenditure and the $n\times 1$ vectors $\{\tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p})\}$ are derived from the $n\times 1$ vectors $\{\boldsymbol{\alpha}_k(\boldsymbol{p})\}$ in (1). Thus, rank can be no more than three and the $\{h_k(m)\}$ are closely related due to symmetry.

Homogeneity and adding up imply further that $f(m) \in \{\ln m, m^\kappa, m^{\iota\tau}\}$, where $\kappa \in \mathbb{R}$, $\tau \in \mathbb{R}_+$, and $\iota = \sqrt{-1}$, with the last case possible only if K=3. Solutions for the full rank

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[4]Also see Russell and Farris (1993, 1998) and Jerison (1993) for more details.

cases have been studied extensively by Gorman (1953, 1961, 1963, 1981), Muellbauer (1975, 1976), Howe, Pollak and Wales (1979), van Daal and Merkies (1989), and most intensively by Lewbel (1987a, 1988, 1989a, 1990). As a result of Lewbel's research program, the indirect preferences of all full rank one and two, and for many full rank three, Gorman systems are known.[5]

Gorman (1981) noted that this class of models is somewhat restricted by the use of nominal income in the $\{h_k(m)\}$. As a result, Lewbel (1989a) introduced the concept of a deflated income Gorman system (hereafter a *Lewbel system*),

$$q_i = \sum_{k=1}^{K} \alpha_{ik}(\boldsymbol{p}) h_k(m/\pi(\boldsymbol{p})), \; i = 1, \ldots, n, \tag{5}$$

where $\pi : \mathbb{R}^n_+ \to \mathbb{R}_+$ is 1° homogeneous. This maintains exact aggregation in deflated income. That is, the real moments of income can be used to estimate aggregate demand functions. Lewbel showed that (5) has maximum rank four, that the indirect utility function for every Lewbel system is equivalent to the indirect utility function for a Gorman system defined over deflated income and conversely, and that the functional form restriction, $f(m) \in \{\ln m, m^{\kappa}, m^{\iota\tau}\}$, is eliminated if we use (5) in place of (1).

While aggregation is an important strength of Gorman and Lewbel systems, *global regularity* often can be a weakness. This class of models is typically economically well-behaved only in a local sense – i.e., in a neighborhood of any point where integrability is satisfied. Only rank one systems can satisfy all of the following on $\mathbb{R}^n_+ \times \mathbb{R}_+$: (a) quantities demanded vanish when income is zero, $\boldsymbol{q}(\boldsymbol{p}, 0) = \boldsymbol{0}$; (b) the expenditure function is

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[5]This includes, *inter alia*, quadratic utility and many extensions, the linear expenditure system, almost ideal demand system (AIDS), PIGL, PIGLOG, and rank three extensions of these models, as well as the Rotterdam, generalized Leontief, normalized quadratic, aggregable translog, and trigonometric demand systems. The implications of symmetry, homogeneity, and adding up on the *demand equations* for all full rank Gorman systems have been identified by Lewbel and van Daal and Merkies. The *indirect preferences* of most full rank three models remains an open question. The solution to this question is one implication of proposition 1 in the next section, which gives an explicit group representation for all Gorman demand systems. Having this solution in hand greatly simplifies the usual tasks of economic analysis, such as welfare measurement, calculating the true cost of living index, and so forth.

strictly increasing in u; and (c) the expenditure function is strictly increasing, 1° homogeneous, and weakly concave in $\boldsymbol{p}$ (Hildenbrand, 2007).[6] Economic regularity – consistency with economic theory – often is a driving force when one specifies an empirical model (Barnett, 2002; Barnett and Lee, 1985; Barnett, Lee and Wolfe, 1985; and Cooper and McLaren 1992, 1996), while homothetic systems are quite restrictive. Hence, it is important to identify flexible, well-behaved demand systems outside of the Gorman class.

This leads to the second approach, which specifies an attractive indirect utility function, $v(\boldsymbol{p}, m)$, and uses Roy's Identity to obtain demands. The properties of economically regular, continuous indirect utility functions are well-known – they are 0° homogeneous in prices and income, decreasing and quasi-convex in prices, and increasing in income. Important, but widely disparate contributions to the development of indirect utility functions include, *inter alia*, the translog (Christensen, Jorgenson and Lau, 1975; Jorgenson, 1990; Jorgenson, Lau and Stoker, 1980, 1981, 1982; Jorgenson and Slesnick, 1984, 1987), the reciprocal indirect generalized Leontief, minflex Laurent and normalized quadratic (Barnett and Lee, 1985; Barnett, Lee and Wolfe, 1985; Diewert and Wales, 1988), the translog, generalized Leontief, and generalized McFadden cost functions (Diewert and Wales, 1987), fractional demand systems (Lewbel 1987b), the normalized quadratic expenditure function (Diewert and Wales, 1988), the modified AIDS (Cooper and McLaren, 1992), the general exponential form (Cooper and McLaren, 1996), and the rational rank four demand system (Lewbel 2003, 2004). However, only the EXP model of Lewbel (1987b) is globally regular and only the rational rank four system of Lewbel (2003, 2004) improves on the rank and flexibility properties of a Gorman system.

This paper presents the solution to integrability for *all* Gorman and Lewbel systems, showing in the process that each is a special case of a projective group transformation (Olver, 1993). This representation allows one to choose a relatively small number of

[6] The most restrictive property is the requirement that quantities demanded vanish identically when income is zero. A large subset of Gorman and Lewbel systems can satisfy (b) and (c) globally – for example, the linear expenditure system and many other quasi-homothetic systems – and all Gorman systems can satisfy these two conditions locally.

functions of prices, income, and the utility index to determine *ex ante* the structure of *any* aggregable demand system. We then present necessary and sufficient conditions for full rank and global regularity of a large class of indirect utility functions. This class typically generates rational demand equations that are not exactly aggregable. But combining these results lets us derive a new class of flexible models to nest these two distinct approaches. This new class of demand models can have arbitrarily large full rank (up to the number of goods in the demand system) and be used to test for aggregation, global regularity, functional form, and flexibility. Finally, we show algebraically how to represent each of the above indirect utility functions as a member of this general, flexible class of indirect preference functions.

2. The Group Structure of Gorman and Lewbel Systems

Recalling equation (4) of the introduction, but dropping the ~'s for notational simplicity, any full rank Gorman system can be written as a special case of the quadratic form[7]

$$\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_1(\boldsymbol{p}) + \boldsymbol{\alpha}_2(\boldsymbol{p})y(\boldsymbol{p},u) + \boldsymbol{\alpha}_3(\boldsymbol{p})y(\boldsymbol{p},u)^2. \tag{6}$$

For example, the full rank 3 QES, extended PIGL, and extended PIGLOG systems of Howe, Pollak and Wales (1979), Lewbel (1987a, 1990) and van Daal and Merkies (1989) all can be written in the compact form (see section one of the Appendix for details),

$$\frac{\partial}{\partial \boldsymbol{p}}\left(\frac{f(e(\boldsymbol{p},u)) - \beta_1(\boldsymbol{p})}{\beta_2(\boldsymbol{p})}\right) = \left[\theta(\beta_3(\boldsymbol{p})) + \left(\frac{f(e(\boldsymbol{p},u)) - \beta_1(\boldsymbol{p})}{\beta_2(\boldsymbol{p})}\right)^2\right]\frac{\partial \beta_3(\boldsymbol{p})}{\partial \boldsymbol{p}}, \tag{7}$$

which is algebraically equivalent to (6) with the following definitions:

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[7]Complete mathematical details for the QES and full rank 3 extended PIGL and PIGLOG cases are presented in the first section of the Appendix. LaFrance, Beatty and Pope (2005) details this property for the full rank three trigonometric model of Lewbel (1988, 1990). An argument for the rank one and two cases is in Lewbel (1988).

$$\boldsymbol{\alpha}_1 = \frac{\partial \beta_1}{\partial \boldsymbol{p}} + \frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \boldsymbol{p}} + \left[\frac{\beta_1^2}{\beta_2} + \theta(\beta_3)\beta_2\right]\frac{\partial \beta_3}{\partial \boldsymbol{p}}; \boldsymbol{\alpha}_2 = -\frac{2\beta_1}{\beta_2}\frac{\partial \beta_3}{\partial \boldsymbol{p}}; \text{and } \boldsymbol{\alpha}_3 = \frac{1}{\beta_2}\frac{\partial \beta_3}{\partial \boldsymbol{p}}. \tag{8}$$

In (6), rank one results when $\boldsymbol{\alpha}_1 \neq \mathbf{0} = \boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_3,$ full rank two when $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \neq \mathbf{0} = \boldsymbol{\alpha}_3,$ and full rank three when $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3 \neq \mathbf{0}.$

Also recall that any Lewbel system can be written as

$$\boldsymbol{q} = \sum\nolimits_{k=1}^{K} \boldsymbol{\alpha}_k(\boldsymbol{p}) h_k\left(e(\boldsymbol{p},u)/\pi(\boldsymbol{p})\right), \tag{9}$$

where $\pi : \mathbb{R}_+^n \to \mathbb{R}_+,$ $\pi \in \mathcal{C}^\infty,$ is increasing, 1° homogeneous, and concave in $\boldsymbol{p}$. To relate Lewbel systems to Gorman systems, first note that adding up implies

$$m \equiv \sum\nolimits_{k=1}^{K} \boldsymbol{p}^\top \boldsymbol{\alpha}_k(\boldsymbol{p}) h_k\left(m/\pi(\boldsymbol{p})\right). \tag{10}$$

As a result, linear independence of the income functions, $\{h_k(x)\},$ implies that one and only one income function must be $m/\pi(\boldsymbol{p})$ and the associated vector of price functions must be $\partial\pi(\boldsymbol{p})/\partial\boldsymbol{p}$. Without loss of generality (WLOG), let this be the first one, and bring it to the left-hand side of (9) to obtain,

$$\boldsymbol{q} - \frac{m}{\pi(\boldsymbol{p})}\frac{\partial \pi(\boldsymbol{p})}{\partial \boldsymbol{p}} = \sum\nolimits_{k=2}^{K} \boldsymbol{\alpha}_k(\boldsymbol{p}) h_k\left(m/\pi(\boldsymbol{p})\right). \tag{11}$$

It follows that

$$\frac{\partial\left(e(\boldsymbol{p},u)/\pi(\boldsymbol{p})\right)}{\partial \boldsymbol{p}} = \sum\nolimits_{k=2}^{K} \tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p}) h_k\left(e(\boldsymbol{p},u)/\pi(\boldsymbol{p})\right), \tag{12}$$

where now $\tilde{\boldsymbol{\alpha}}_k \equiv \boldsymbol{\alpha}_k/\pi$, $k = 2, \cdots, K.$ This has the structure of a Gorman system, but one in which symmetry is the only issue since homogeneity is incorporated automatically by deflating income and the implications of adding up are satisfied.[8] Hence, in a full rank

[8]Note, however, that adding up also implies $\boldsymbol{p}^\top \tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p}) = 0,$ $k = 2, \cdots, K.$

Lewbel system, applying Lie (1880) yields the quadratic structure (6), but now with $y(\boldsymbol{p},u) = f(e(\boldsymbol{p},u)/\pi(\boldsymbol{p}))$.

Differential equations of the form (6) are called Ricatti equations and have been studied extensively in the mathematical theory of differential equations. Closed form solutions do not exist except in special cases. However, in section three of the Appendix we show that all solutions to (6) are characterized by the function, $w:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $w\in\mathcal{C}^\infty$, defined by

$$w(\beta_3(\boldsymbol{p}),u) = \begin{cases} u, & \text{if } K = 1,2 \text{ or } K = 3 \text{ and } \theta'(x) = 0, \\ u + \int_0^{\beta_3(\boldsymbol{p})} [\theta(x) + w(x,u)^2] dx, & \text{if } K = 3 \text{ and } \theta'(x) \neq 0, \end{cases} \tag{13}$$

subject to $w(0,u) = u$ and $\partial w(0,u)/\partial x = \theta(0) + u^2$, for some $\beta_3:\mathbb{R}_+^n\to\mathbb{R}$, $\beta_3\in\mathcal{C}^\infty$, and some $\theta:\mathbb{R}\to\mathbb{R}$, $\theta\in\mathcal{C}^\infty$.[9,10] With this background, we now present a complete group characterization of all full rank Gorman and Lewbel demand systems (a proof of this proposition is presented in the third section of the Appendix).

Proposition 1: *Let* $\pi:\mathbb{R}_+^n\to\mathbb{R}_+$, $\pi\in\mathcal{C}^\infty$, *be positive-valued, 1° homogeneous, increasing, and concave in* $\boldsymbol{p}$*; let* $\eta:\mathbb{R}_+^n\to\mathbb{R}$, $\eta\in\mathcal{C}^\infty$, *be 0° homogeneous in* $\boldsymbol{p}$*; let* $\alpha,\beta,\gamma,\delta:\mathbb{R}_+^n\to\mathbb{C}=\{x+\imath y,\ x,y\in\mathbb{R}\}$, $\alpha,\beta,\gamma,\delta\in\mathcal{C}^\infty$, *be 0° homogeneous and satisfy the normalizing identity* $\alpha\delta-\beta\gamma\equiv 1$*; and let* $f:\mathbb{R}\to\mathbb{C}$, $f\in\mathcal{C}^\infty$,

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[9]The first line on the right-hand side of (13) is a normalization of the utility index that can be made without loss of generality whenever $K = 1,2$ or $K = 3$ and $\theta(x) = \lambda$, a constant. See section three of the Appendix for complete details.

[10]In general, $\theta'(x)\neq 0$ "… complicates the demand equations while adding nothing to either income or price flexibility, so demands with [$\theta(x)\neq\lambda$] are not likely to be of much practical interest." (Lewbel, 1987a: p. 1454). This argument is repeated in Lewbel (1990: p. 292). While we agree with this assessment, the complete class of Gorman and Lewbel demand systems includes all $\theta:\mathbb{R}\to\mathbb{R}$, and this is a substantial increase in the class of indirect preferences for aggregable demand systems.

$f' > 0$. *Then the expenditure function for any full rank Gorman or Lewbel system is a special case of:*

$$f\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right) = \frac{\alpha(\boldsymbol{p})w(\eta(\boldsymbol{p}),u)+\beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})w(\eta(\boldsymbol{p}),u)+\delta(\boldsymbol{p})}. \tag{14}$$

The necessity part of the proof is quite involved, and is relegated to the Appendix. But it is constructive at this point to consider sufficiency. This can be shown by simply differentiating (14). To make the notation as compact as possible, let a bold subscript $\boldsymbol{p}$ denote a vector of partial derivatives with respect to prices, suppress the arguments of the price functions, and make the substitution $m = e(\boldsymbol{p},u)$ throughout, to yield (after considerable straightforward, but very tedious, algebra):

$$\begin{aligned}\boldsymbol{q} = \left(\pi_{\boldsymbol{p}}/\pi\right)m + \frac{\pi}{f'(m/\pi)}\Big\{&\left[\alpha\beta_{\boldsymbol{p}} - \beta\alpha_{\boldsymbol{p}} + (\alpha^2\theta+\beta^2)\eta_{\boldsymbol{p}}\right] \\ &+\left[\beta\gamma_{\boldsymbol{p}} - \gamma\beta_{\boldsymbol{p}} + \delta\alpha_{\boldsymbol{p}} - \alpha\delta_{\boldsymbol{p}} - 2(\alpha\gamma\theta+\beta\delta)\eta_{\boldsymbol{p}}\right] f(m/\pi) \\ &+\left[\gamma\delta_{\boldsymbol{p}} - \delta\gamma_{\boldsymbol{p}} + (\alpha^2\theta+\delta^2)\eta_{\boldsymbol{p}}\right] f(m/\pi)^2\Big\}.\end{aligned} \tag{15}$$

First, note that there are a total of four income terms on the right-hand side of (15) with four associated vectors of price functions that (in principle) can be linearly independent. However, if $f(x) \in \{\ln x, x^{\kappa}, x^{\iota\tau}\}$, then either $f(x)/f'(x)$ or $1/f'(x)$ is proportional to *x*. Since the first term on the right-hand side is proportional to $x = m/\pi$, any of these choices for the functional form for *f* reduces the number of independent income terms by one. Thus, a Lewbel system has rank equal to one plus the rank of an otherwise identical Gorman system if and only if $f(x) \notin \{\ln x, x^{\kappa}, x^{\iota\tau}\}$, i.e., it is *not* one of the functional forms found by Gorman (1981). This result shows precisely how rank increases by one vector of price functions and one linearly independent income function in a Lewbel system.

Readers familiar with the theory of Lie groups will recognize the right-hand side of (14) as a projective group transformation from *w* to *f*. This group is associated with the group of 2×2 matrices with a unit determinant:

$$\boldsymbol{A} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \ \alpha\delta - \beta\gamma = 1.$$

We discuss the nature of this group, known as the special linear group two, $\mathfrak{sl}(2)$, in some detail in section two of the Appendix. Restrictions on $\{\alpha, \beta, \gamma, \delta, \eta\}$ generate the various ranks. In rank one, $\alpha = \delta = 1$ and $\beta = \gamma = \eta = 0$. For rank two, the nature of the restrictions depends on the functional form of $f(x)$.[11] If $f(x) = \ln x$ (the PIGLOG model), $\alpha = 1/\delta$ and $\beta = \gamma = \eta = 0$, while if $f(x) \neq \ln x$ (all other cases, including the PIGL model), $\alpha = \delta = 1$, β is unrestricted, and $\gamma = \eta = 0$. In rank three and four systems, there are two and three independent price indices on the right-hand side, respectively, and the algebraic relationships among them can be complicated. Nevertheless, these results give the following simple rule for the rank of any Gorman or Lewbel system.

> ***Corollary 1:*** *The rank of a full rank Gorman or Lewbel system is the number of unrestricted 0° homogeneous price indices on the right-hand side of (14), plus one to account for the 1° homogeneous deflator on the left-hand side.*

4. Directly Specified Indirect Utility Functions

The consumer's indirect utility function, $v(\boldsymbol{p}, m) = \max_{\boldsymbol{q} \geq \boldsymbol{0}} \{u(\boldsymbol{q}) : \boldsymbol{p}^\top \boldsymbol{q} = m\}$, is *economically regular* if it is continuous, 0° homogeneous in prices and income, decreasing and quasi-convex in prices, and increasing in income. The simplest way to directly specify a function with these properties on $\mathbb{R}^n_{++} \times \mathbb{R}_{++}$ is the homothetic case, $v(\boldsymbol{p}, m) = m/\pi(\boldsymbol{p})$, where $\pi : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ is strictly positive valued, increasing, weakly concave, and 1° homogeneous. Due to the ordinal nature of the utility index, we can always normalize the homothetic case this way. By Lemma 2 in the Appendix of Cooper and McLaren (2006)

[11] For rank one, the functional form of $f(x)$ is unrestricted due to the ordinal nature of the utility index. Also recall from (13) that in both the rank one and two cases, $w(\eta(\boldsymbol{p}), u) = u$.

we also know that $v(\boldsymbol{p},m)=m/\pi(\boldsymbol{p})$ is convex in $\boldsymbol{p}$. Since a convex function of a convex function is convex, $\tilde{v}(\boldsymbol{p},m)=\varphi\left(m/\pi(\boldsymbol{p})\right),\ \varphi'>0,\ \varphi''\geq 0,$ also satisfies the above regularity conditions throughout $\mathbb{R}^n_{++}\times\mathbb{R}_{++}$. This extends to several terms by the fact that a sum of convex functions is convex, as stated in the following.[12]

> ***Proposition 2.*** *Given the J positive-valued, increasing, concave, 1° homogeneous, and linearly independent functions,* $\pi_j:\mathbb{R}^n_{++}\to\mathbb{R}_{++},\ \pi_j\in\mathcal{C}^\infty,\ j=1,\cdots,J,$ *the J increasing and linearly independent functions,* $\varphi_j:\mathbb{R}_{++}\to\mathbb{R},\ \varphi_j\in\mathcal{C}^\infty,\ \varphi_j'>0,$ $j=1,\cdots,J,$ *and the increasing function,* $f:\mathbb{R}\to\mathbb{R},\ f\in\mathcal{C}^\infty,\ f'>0,$ *the indirect utility function*
>
> $$v(\boldsymbol{p},m)=f\left(\sum_{j=1}^J\varphi_j\left(m/\pi_j(\boldsymbol{p})\right)\right), \tag{16}$$
>
> *is globally regular* $\forall\ (\boldsymbol{p},m)\in\mathbb{R}^n_{++}\times\mathbb{R}_{++}$ *if* $\varphi_j''\geq 0\ \forall\ j,$ *and full rank if and only if at most one* $\varphi_j''=0$.

Proof: Sufficiency is obvious. For necessity, apply Roy's Identity to (16), which gives

$$\boldsymbol{q}=\frac{\sum_{j=1}^J\left(\pi_{j\boldsymbol{p}}(\boldsymbol{p})/\pi_j(\boldsymbol{p})^2\right)m\varphi_j'(m/\pi_j(\boldsymbol{p}))}{\sum_{j=1}^J\varphi_j'(m/\pi_j(\boldsymbol{p}))/\pi_j(\boldsymbol{p})}, \tag{17}$$

[12]This clearly is not the only way to obtain high rank, regular, rational demand systems. One very recent alternative is the approach of Cooper and McLaren (2006), who define an *effectively globally regular* demand system as one in which preferences are economically regular for all income levels above a threshold, which in turn is a function of prices. They develop a new class of demand systems based on the ratio of two functions with certain regularity properties. Members of this class can have arbitrarily high rank (up to the number of commodities) in the sense of Lewbel (1991). McLaren and Wong (2007) extend these results to a new class of composite direct, inverse, or mixed demand systems whose members are effectively globally regular and also can achieve arbitrarily high rank.

In the demands in (17), rank is defined by the number of linearly independent vectors, $\pi_{j\boldsymbol{p}} / \pi_j^2$, $j = 1, \cdots, J$, as long as the income functions $\{m\varphi'_j\}$ are linearly independent (Lewbel, 1990). The latter property implies that at most one $\varphi''_j = 0$. Otherwise reduce J by one through an algebraic combination of two or more of the φ_j's without any change in the demand equations. Hence at least J–1 of the $\{\varphi_j\}$ must have nonzero curvature in a full rank system. ■

To clarify, f only needs to be increasing because the utility index is ordinal. But in the next section, we nest this class of indirect preferences with full rank Gorman and Lewbel systems. There the curvature of f plays a role in both the rank and the region of regularity of the system. Second, we could extend (16) by adding a 0° homogeneous function of prices, $\varphi_{J+1} : \mathbb{R}^n_{++} \to \mathbb{R}$, $\varphi_{J+1} \in \mathcal{C}^\infty$, to the argument of f on the right-hand side of (16) and still (at least potentially) have a full rank system. Some existing indirect utility functions have this property – the translog of Christensen, Jorgenson, and Lau (1975) and the rational rank four system of Lewbel (2003, 2004) are two important examples. However, this modification can negatively affect the rank or the region of regularity of the demands. Finally, since the sum of quasi-convex functions is not quasi-convex in general (Arrow and Enthoven, 1961), each of the $\{\varphi_j\}$ must be at least weakly convex if we want to be sure that the indirect utility function is globally regular for this class of models.

5. Full Rank Demand Systems with Rank > 4

The previous two sections presented characterizations for two distinct approaches to the specification of empirical demand models. However, if we interchange the left- and right-hand sides of (16), and again make the substitution $m = e(\boldsymbol{p}, u)$, we obtain

$$f\left(\sum_{j=1}^{J} \varphi_j \left(e(\boldsymbol{p}, u) / \pi_j(\boldsymbol{p})\right)\right) = u. \qquad (18)$$

This class of indirect preference models has the form of a rank one Gorman system on the right-hand side (i.e., $\alpha = \delta = 1$ and $\beta = \gamma = \eta = 0$). It follows that we can extend this to the complete Gorman and Lewbel class by replacing the right-hand side of (18) with the right-hand side of (14). This extension generates a large and flexible class of demand models that can have full rank up to $J+3 \leq n$ while nesting the complete class of exactly aggregable Gorman and Lewbel systems of proposition 1 with the above class of globally regular demand systems of proposition 2 as special cases.[13]

Proposition 3: *Given the assumptions and definitions for proposition 1 and 2, the expenditure function defined implicitly by*

$$f\left(\sum_{j=1}^{J}\varphi_j\left(e(\boldsymbol{p},u)/\pi_j(\boldsymbol{p})\right)\right)=\frac{\alpha(\boldsymbol{p})w(\eta(\boldsymbol{p}),u)+\beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})w(\eta(\boldsymbol{p}),u)+\delta(\boldsymbol{p})}, \tag{19}$$

has full rank J, J+1, J+2, or J+3 according to corollary 1 if and only if the unrestricted elements of $\{\alpha,\beta,\gamma,\delta,\eta\}$ *are linearly independent of the elements of* $\{\pi_1,\cdots,\pi_J\}$ *and no more than one of* $\{\varphi_1'',\cdots,\varphi_J'', f''\}$ *vanish.*

Proof: Given earlier results, both the necessity and the sufficiency of this result can be demonstrated by simply applying Hotelling's/Shephard's Lemma, which yields

$$\boldsymbol{q}=\frac{m\sum_{j=1}^{J}\varphi_j'\,\pi_{j\boldsymbol{p}}\big/\pi_j^2}{\sum_{j=1}^{J}\varphi_j'\big/\pi_j}+\frac{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2 y+\boldsymbol{\alpha}_3 y^2}{f'\times\sum_{j=1}^{J}\varphi_j'\big/\pi_j}, \tag{20}$$

with $\boldsymbol{\alpha}_1=\alpha\beta_{\boldsymbol{p}}-\beta\alpha_{\boldsymbol{p}}+(\alpha^2\theta+\beta^2)\eta_{\boldsymbol{p}}$, $\boldsymbol{\alpha}_2=\beta\gamma_{\boldsymbol{p}}-\gamma\beta_{\boldsymbol{p}}+\delta\alpha_{\boldsymbol{p}}-\alpha\delta_{\boldsymbol{p}}-2(\alpha\gamma\theta+\beta\delta)\eta_{\boldsymbol{p}}$, and $\boldsymbol{\alpha}_3=\gamma\delta_{\boldsymbol{p}}-\delta\gamma_{\boldsymbol{p}}+(\alpha^2\theta+\delta^2)\eta_{\boldsymbol{p}}$. ■

To clarify, if J=1 we obtain the set of full rank Gorman and Lewbel demand systems because the composite function $f\circ\varphi_1:\mathbb{R}_+\to\mathbb{R}$ is strictly monotonic. Conversely, if we

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[13]Alternatively, one could substitute the class of indirect utility functions in Cooper and McLaren (2006) or McLaren and Wong (2007) on the left-hand-side of (19), generating a class of rational demand models to nest the set of *exactly aggregable* systems with the set of *essentially globally regular* systems, rather than the set of *globally regular* systems presented in the previous section.

restrict the right-hand side of (19) such that $\alpha = \delta = 1$ and $\beta = \gamma = \eta = 0,$ we obtain the set of globally regular full rank rational demand systems of proposition 2. Finally, if $f'' = 0,$ then the denominators of the two fractions on the right-hand side of (20) are proportional. In that case, the system can not have full rank unless $\varphi_j'' \neq 0 \ \forall \ j.$ Otherwise, at least one term in the numerator of the first ratio is proportional to the corresponding term in the numerator of the second ratio for which φ_j' is constant. Conversely, if $f'' \neq 0,$ then at most one $\varphi_j'' = 0$ if the system has full rank. Otherwise two income terms in the numerator of the first ratio are proportional and we can reduce *J* with an algebraic combination of the price and income functions. Thus, at least *J* of $\{\varphi_1, \cdots, \varphi_J, f\}$ must have nonzero curvature for the system to have full rank.

We can illustrate this result with a simple example based on *J* polynomials in income. Define $\varphi_j(m/\pi_j(\boldsymbol{p})) = (\phi_j/j) \times (m/\pi_j(\boldsymbol{p}))^j,\ j = 1, \cdots, J,$ and $f(x) = e^x,$ to yield:

$$\boldsymbol{q} = \frac{\sum_{j=1}^{J} \left[\phi_j \times \pi_{\boldsymbol{jp}} \times (m/\pi_j)^{j+1} \right] + m(\boldsymbol{\alpha}_1 y^{-1} + \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3 y)}{\sum_{j=1}^{J} \phi_j (m/\pi_j)^j}. \tag{21}$$

The numerator is a J+1$^{\text{st}}$–order polynomial in income plus the sum of the linearly independent terms my^{-1} and my. The denominator is a J^{th}–order polynomial in income and the rank is *J*+3. Other choices for $\{\varphi_1, \cdots, \varphi_J, f\}$ yield a limitless set of similar results, so long as at least *J* have nonzero curvature.

The fourth section of the Appendix has the derivations needed to express these models as special cases of (16): the reciprocal indirect generalized Leontief and minflex Laurent of Barnett and Lee (1985) and Barnett, Lee and Wolfe (1985); the LOG2 and EXP fractional demand systems of Lewbel (1987b); the reciprocal indirect normalized quadratic of Diewert and Wales (1988); the modified AIDS (MAIDS) of Cooper and McLaren (1992); and the general exponential form (GEF) of Cooper and McLaren (1996). It also has the derivations needed to express the translog of Christenson, Jorgenson and Lau (1975), the TAN model of Lewbel (1987b), and the rational rank four demand system of Lewbel (2003, 2004) as special cases of (19). This includes virtually all existing alternatives to the Gorman and Lewbel class of demand systems. Among this large number of alternatives, the rational rank four demand system is full rank four, the minflex Laurent is reduced rank three, the normalized quadratic is reduced rank two, all of the others are full rank two, and only the EXP model can satisfy proposition 2.

6. Conclusions

Common reasons for the choice of functional form for demand analysis include parsimony, ease of estimation and interpretation, generality, flexibility, aggregation, and consistency with economic theory. Since the path-breaking papers of Gorman, flexibility and aggregation have guided much of the development and application of applied demand analysis. The rank of Engel curves is a central feature of this research. The recent emphasis on household level data, empirical models, and methods appears to have somewhat decreased the emphasis on aggregation in empirical work. The availability of large household level data sets also allows for increased parametric flexibility, and this added flexibility is increasingly less costly to implement with modern computation and storage capacities. Nevertheless, it is routine to impose the theoretical properties associated with Slutsky symmetry and negativity, homogeneity, and adding up.

This paper has developed a flexible structural model of micro-level behavior that can be used to nest and test models that easily can be made to be globally regular with models that easily can be made to be aggregable. In addition to all Gorman and Lewbel systems of aggregable demand equations, we have shown that almost all common alternatives are encompassed in this new class of rational demand systems, although none of these alternatives improve on the rank and flexibility properties of a Lewbel system, and only one is globally economically regular. Thus, we have reframed and greatly extended a large and influential literature on functional form, flexibility, regularity, and aggregation in applied economic analysis.

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Appendix

A.1 Representation Algebra

Assume that the expenditure function, $e:\mathbb{R}^n_{++}\times\mathbb{R}\to\mathbb{R}_{++}$, defined by

$$e(\boldsymbol{p},u)\equiv\min_{\boldsymbol{q}\in\mathbb{R}^n_+}\left\{\boldsymbol{p}^\top\boldsymbol{q}:u(\boldsymbol{q})\geq u\right\}, \tag{A.1}$$

is smooth, $e\in\mathcal{C}^\infty$, increasing, 1° homogeneous, and concave in $\boldsymbol{p}$, and increasing in u. This section presents the algebraic steps that will put each of the extended PIGL, PIGLOG, and QES full rank three nominal income Gorman systems in the form,

$$\frac{\partial}{\partial\boldsymbol{p}}\left(\frac{f(e(\boldsymbol{p},u))-\beta_1(\boldsymbol{p})}{\beta_2(\boldsymbol{p})}\right)=\left[\theta(\beta_3(\boldsymbol{p}))+\left(\frac{f(e(\boldsymbol{p},u))-\beta_1(\boldsymbol{p})}{\beta_2(\boldsymbol{p})}\right)^2\right]\frac{\partial\beta_3(\boldsymbol{p})}{\partial\boldsymbol{p}}, \tag{A.2}$$

where $f(m)\in\{\ln m,m^\kappa\}$, $\beta_1,\beta_2,\beta_3:\mathbb{R}^n_{++}\to\mathbb{R}$, $\theta:\mathbb{R}\to\mathbb{R}$, and $\beta_1,\beta_2,\beta_3,\theta\in\mathcal{C}^\infty$. With the change of variables to $z(\boldsymbol{p},u)=\left[f(e(\boldsymbol{p},u))-\beta_1(\boldsymbol{p})\right]/\beta_2(\boldsymbol{p})$, this then simplifies to

$$\partial z(\boldsymbol{p},u)/\partial\boldsymbol{p}=\left[\theta(\beta_3(\boldsymbol{p}))+z(\boldsymbol{p},u)^2\right]\partial\beta_3(\boldsymbol{p})/\partial\boldsymbol{p}, \tag{A.3}$$

which is useful for characterizing the complete class of solutions for the indirect preferences of these models. Throughout this section, a bold subscript $\boldsymbol{p}$ denotes a vector of partial derivatives with respect to prices, we use notation consistent with (A.2) to replace the corresponding notation in the original articles, and often omit the arguments of functions for compactness.

In van Daal and Merkies (1989), equation (2), group terms in $\beta_2(\boldsymbol{p})^{-1}$:

$$\boldsymbol{q}=\beta_2^{-1}\left(m^2\beta_{3\boldsymbol{p}}+m\beta_{2\boldsymbol{p}}-2m\beta_1\beta_{3\boldsymbol{p}}+\beta_1^2\beta_{3\boldsymbol{p}}-\beta_1\beta_{2\boldsymbol{p}}\right)+\beta_{1\boldsymbol{p}}+\theta(\beta_3)\beta_2\beta_{3\boldsymbol{p}}. \tag{A.4}$$

where $\boldsymbol{p}$ subscripts denote differentiation. Regroup terms in the parentheses:

$$\boldsymbol{q}=\beta_2^{-1}\left[(m-\beta_1)^2\beta_{3\boldsymbol{p}}+(m-\beta_1)\beta_{2\boldsymbol{p}}\right]+\beta_{1\boldsymbol{p}}+\theta(\beta_3)\beta_2\beta_{3\boldsymbol{p}}. \tag{A.5}$$

Gather terms in $\beta_{3\boldsymbol{p}}$, divide both sides by β_2, and isolate $\beta_{3\boldsymbol{p}}$ on the right:

$$\frac{\boldsymbol{q}-\beta_{1\boldsymbol{p}}}{\beta_2}-\frac{(m-\beta_1)\beta_{2\boldsymbol{p}}}{\beta_2^2}=\left[\left(\frac{m-\beta_1}{\beta_2}\right)^2+\theta(\beta_3)\right]\beta_{3\boldsymbol{p}}\,. \qquad \text{(A.6)}$$

To obtain (A.2), note that

$$\frac{\partial}{\partial\boldsymbol{p}}\left(\frac{e(\boldsymbol{p},u)-\beta_1(\boldsymbol{p})}{\beta_2(\boldsymbol{p})}\right)=\frac{(\boldsymbol{q}-\beta_{1\boldsymbol{p}}(\boldsymbol{p}))}{\beta_3(\boldsymbol{p})}-\frac{(e(\boldsymbol{p},u)-\beta_1(\boldsymbol{p}))\beta_{3\boldsymbol{p}}(\boldsymbol{p})}{\beta_3(\boldsymbol{p})^2}\,. \qquad \text{(A.7)}$$

In Lewbel (1990), case iv, move $\tau m^{\tau-1}$ to the left-hand side, define $\tilde{\beta}_2(\boldsymbol{p})\equiv\beta_2(\boldsymbol{p})^{1/\tau}$ and $\tilde{\beta}_1(\boldsymbol{p})\equiv\beta_1(\boldsymbol{p})/\beta_2(\boldsymbol{p})$:

$$\begin{aligned}\tau m^{\tau-1}\boldsymbol{q}&=\tilde{\beta}_2^{\tau}\tilde{\beta}_{1\boldsymbol{p}}+\tilde{\beta}_1^2\tilde{\beta}_2^{\tau}\beta_{3\boldsymbol{p}}+\theta(\beta_3)\tilde{\beta}_2^{\tau}\beta_{3\boldsymbol{p}}\\&\quad+\left(\frac{\tau\tilde{\beta}_{2\boldsymbol{p}}}{\tilde{\beta}_2}-2\tilde{\beta}_1\beta_{3\boldsymbol{p}}\right)m^{\tau}+\frac{\beta_{3\boldsymbol{p}}}{\tilde{\beta}_2^{\tau}}m^{2\tau}.\end{aligned} \qquad \text{(A.8)}$$

Group terms in $\beta_{3\boldsymbol{p}}$ and $\tilde{\beta}_2^{\tau}$:

$$\begin{aligned}\tau m^{\tau-1}\boldsymbol{q}&=\tilde{\beta}_2^{\tau}\left\{\left[\left(\frac{m^{2\tau}}{\tilde{\beta}_2^{2\tau}}\right)-2\tilde{\beta}_1\left(\frac{m^{\tau}}{\tilde{\beta}_2^{\tau}}\right)+\tilde{\beta}_1^2+\theta(\beta_3))\right]\beta_{3\boldsymbol{p}}+\tilde{\beta}_{1\boldsymbol{p}}\right\}+\tau\frac{\tilde{\beta}_{2\boldsymbol{p}}}{\tilde{\beta}_2}m^{\tau}\\&=\tilde{\beta}_2^{\tau}\left\{\left[\left(\frac{m^{\tau}-\tilde{\beta}_1\tilde{\beta}_2^{\tau}}{\tilde{\beta}_2^{\tau}}\right)^2+\theta(\beta_3)\right]\beta_{3\boldsymbol{p}}+\tilde{\beta}_{1\boldsymbol{p}}\right\}+\tau\frac{\tilde{\beta}_{2\boldsymbol{p}}}{\tilde{\beta}_2}m^{\tau}.\end{aligned} \qquad \text{(A.9)}$$

Isolate the terms in $\beta_{3\boldsymbol{p}}$ on the right:

$$\frac{\tau m^{\tau-1}\boldsymbol{q}}{\tilde{\beta}_2^{\tau}}-\frac{\tau m^{\tau}\tilde{\beta}_{2\boldsymbol{p}}}{\tilde{\beta}_2^{\tau+1}}-\tilde{\beta}_{1\boldsymbol{p}}=\left[\left(\frac{m^{\tau}-\tilde{\beta}_1\tilde{\beta}_2^{\tau}}{\tilde{\beta}_2^{\tau}}\right)^2+\theta(\beta_3)\right]\beta_{3\boldsymbol{p}}\,. \qquad \text{(A.10)}$$

Note that

$$\frac{\partial}{\partial\boldsymbol{p}}\left(\frac{e(\boldsymbol{p},u)^{\tau}}{\tilde{\beta}_2(\boldsymbol{p})^{\tau}}-\tilde{\beta}_1(\boldsymbol{p})\right)=\frac{\tau e(\boldsymbol{p},u)^{\tau-1}\boldsymbol{q}}{\tilde{\beta}_2^{\tau}(\boldsymbol{p})}-\frac{\tau e(\boldsymbol{p},u)^{\tau}\tilde{\beta}_{2\boldsymbol{p}}(\boldsymbol{p})}{\tilde{\beta}_2^{\tau+1}(\boldsymbol{p})}-\tilde{\beta}_{1\boldsymbol{p}}(\boldsymbol{p})\,. \qquad \text{(A.11)}$$

Substitute $\beta_1(\boldsymbol{p}) \equiv \tilde{\beta}_1(\boldsymbol{p})\tilde{\beta}_2(\boldsymbol{p})^{\tau}$ and $\beta_2(\boldsymbol{p}) \equiv \tilde{\beta}_2(\boldsymbol{p})^{\tau}$ in (A.10) and (A.11) to get (A.2).

In Lewbel (1990), case v, fix a sign error and a typographical error on the right (see Lewbel, 1990, p.297) and move 1/*m* to the left-hand side:

$$\begin{aligned}\frac{\boldsymbol{q}}{m} = \frac{\beta_{3\boldsymbol{p}}}{\beta_2}(\ln\beta_1)^2 - \frac{\beta_{2\boldsymbol{p}}}{\beta_2}\ln\beta_1 + \frac{\beta_{1\boldsymbol{p}}}{\beta_1} + \theta(\beta_3)\beta_2\beta_{3\boldsymbol{p}} \\ + \left(\frac{\beta_{2\boldsymbol{p}} - 2\beta_{3\boldsymbol{p}}\ln\beta_1}{\beta_2}\right)\ln m + \frac{\beta_{3\boldsymbol{p}}}{\beta_2}(\ln m)^2.\end{aligned} \tag{A.12}$$

Group terms in $\beta_{3\boldsymbol{p}}$:

$$\frac{\boldsymbol{q}}{m} = \beta_2\left[\left(\frac{\ln(m/\beta_1)}{\beta_2}\right)^2 + \theta(\beta_3)\right]\beta_{3\boldsymbol{p}} + \frac{\beta_{1\boldsymbol{p}}}{\beta_1} + \frac{\ln(m/\beta_1)\beta_{2\boldsymbol{p}}}{\beta_2}. \tag{A.13}$$

Isolate the terms involving $\beta_{3\boldsymbol{p}}$ on the right-hand side:

$$\frac{\boldsymbol{q}}{\beta_2 m} - \frac{\beta_{1\boldsymbol{p}}}{\beta_1\beta_2} - \frac{\ln(m/\beta_1)\beta_{2\boldsymbol{p}}}{\beta_2^2} = \left[\left(\frac{\ln(m/\beta_1)}{\beta_2}\right)^2 + \theta(\beta_3)\right]\beta_{3\boldsymbol{p}}. \tag{A.14}$$

To obtain (A.2), note that

$$\frac{\partial}{\partial\boldsymbol{p}}\left(\frac{\ln\left[\dfrac{e(\boldsymbol{p},u)}{\beta_1(\boldsymbol{p})}\right]}{\beta_2(\boldsymbol{p})}\right) = \frac{\boldsymbol{q}}{\beta_2(\boldsymbol{p})e(\boldsymbol{p},u)} - \frac{\beta_{1\boldsymbol{p}}(\boldsymbol{p})}{\beta_1(\boldsymbol{p})\beta_2(\boldsymbol{p})} - \frac{\ln\left[\dfrac{e(\boldsymbol{p},u)}{\beta_1(\boldsymbol{p})}\right]\beta_{2\boldsymbol{p}}(\boldsymbol{p})}{\beta_2^2(\boldsymbol{p})}. \tag{A.15}$$

Therefore, all full rank three QES, extended PIGL, and extended PIGLOG models are characterized by the following separability property.

Lemma 1: *Let* $z:\mathbb{R}^n_{++}\times\mathbb{R}\to\mathbb{R}$, $\theta:\mathbb{R}\to\mathbb{R}$, *and* $\beta_3:\mathbb{R}^n_{++}\to\mathbb{R}$, $z,\theta,\beta_3\in\mathcal{C}^{\infty}$, *satisfy (A.3) with* $\partial\beta_3(\boldsymbol{p})/\partial\boldsymbol{p}\neq\mathbf{0}$, *then* $z(\boldsymbol{p},u)\equiv w(\beta_3(\boldsymbol{p}),u)$, *with* $w(x,u)$ *satisfying the partial differential equation* $\partial w(x,u)/\partial x = \theta(x) + w(x,u)^2$.

Proof: Differentiating both sides of the system of partial differential equations,

$$\frac{\partial^2 z(\boldsymbol{p},u)}{\partial \boldsymbol{p}\partial \boldsymbol{p}^\top} = \theta'(\beta_3(\boldsymbol{p}))\frac{\partial \beta_3(\boldsymbol{p})}{\partial \boldsymbol{p}}\frac{\partial \beta_3(\boldsymbol{p})}{\partial \boldsymbol{p}^\top} + \left[\theta(\beta_3(\boldsymbol{p})) + z(\boldsymbol{p},u)^2\right]\frac{\partial^2 \beta_3(\boldsymbol{p})}{\partial \boldsymbol{p}\partial \boldsymbol{p}^\top} + 2z(\boldsymbol{p},u)\frac{\partial z(\boldsymbol{p},u)}{\partial \boldsymbol{p}}\frac{\partial \beta_3(\boldsymbol{p})}{\partial \boldsymbol{p}^\top}. \tag{A.16}$$

Hence, $(\partial z/\partial \boldsymbol{p}) \times (\partial \beta_3/\partial \boldsymbol{p})^\top$ is symmetric, so that $z(\boldsymbol{p},u) = w(\beta_3(\boldsymbol{p}),u)$ (Goldman and Uzawa 1964, Lemma 1). Differentiating with respect to prices then yields,

$$\frac{\partial z(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \frac{\partial w(\beta_3(\boldsymbol{p}),u)}{\partial \beta_3}\cdot\frac{\partial \beta_3(\boldsymbol{p})}{\partial \boldsymbol{p}} = \left[\theta(\beta_3(\boldsymbol{p})) + w(\beta_3(\boldsymbol{p}),u)^2\right]\frac{\partial \beta_3(\boldsymbol{p})}{\partial \boldsymbol{p}}. \qquad \blacksquare$$

As stated in the main paper, $w:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $w \in \mathcal{C}^\infty$, can be defined implicitly by

$$w(\beta_3(\boldsymbol{p}),u) = \begin{cases} u, & \text{if } K = 1,2 \text{ or } K = 3 \text{ and } \theta'(x) = 0, \\ u + \int_0^{\beta_3(\boldsymbol{p})}\left[\theta(x) + w(x,u)^2\right]dx, & \text{if } K = 3 \text{ and } \theta'(x) \neq 0, \end{cases} \tag{A.17}$$

with the initial conditions $w(0,u) = u$ and $\partial w(0,u)/\partial x = \theta(0) + u^2$.[14] In the third section of this Appendix, we use (13) to obtain the projective group transformations for this class of demand systems. The special cases that can be expressed in terms of elementary functions illustrate the nature of these transformations.

Finally, to complete the full rank three cases, Lewbel (1988, 1990) finds the indirect preferences for the trigonometric full rank three Gorman system,

$$v(\boldsymbol{p},m) = \beta_2(\boldsymbol{p}) + \frac{\beta_3(\boldsymbol{p})\cos\left(\tau \ln(m/\beta_1(\boldsymbol{p}))\right)}{\left[1 - \sin\left(\tau \ln(m/\beta_1(\boldsymbol{p}))\right)\right]}. \tag{A.18}$$

[14] Make the change of variables $w(x,u) = -\partial v(x,u)/\partial x / v(x,u)$ to convert the Riccati partial differential equation in *w* to the linear second-order partial differential equation $\partial^2 v(x,u)/\partial x^2 + \theta(x)v(x,u) = 0$, which requires two initial conditions. The two chosen here normalize the utility index and guarantee smoothness at *x*=0 for all *u*. In general, linear second-order differential equations with non-constant coefficients do not have solutions that are expressible in terms of elementary functions, although solutions in terms of convergent infinite series can often be found (e.g., Boyce and DiPrima 1977, chapter 4).

In the third section of this Appendix, we use this expression to find the projective group transformation for this system. Before proceeding with this, however, the next section discusses the mathematical properties of projective transformation groups.

A.2 Projective Transformation Groups

A *group* is a set of elements that is closed under a binary operator, called *multiplication* regardless of what the operator truly is, an inverse operator, and a well-defined identity operator. The *projective transformation group* is equivalent to the *special linear group two*, $\mathfrak{sl}(2)$. The latter group is defined by the set of all 2×2 matrices,

$$\boldsymbol{A} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \tag{A.19}$$

with a unit determinant, $\alpha\delta - \beta\gamma = 1$ (Olver, 1993). The inverse of any $\boldsymbol{A} \in \mathfrak{sl}(2)$,

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}, \tag{A.20}$$

satisfies $\boldsymbol{A}^{-1} \in \mathfrak{sl}(2)$, since $|\boldsymbol{A}^{-1}| = 1/|\boldsymbol{A}| = 1$. It is clear that the 2×2 identity matrix belongs to $\mathfrak{sl}(2)$, while $\boldsymbol{I}_2\boldsymbol{A} = \boldsymbol{A}$ for all $\boldsymbol{A} \in \mathfrak{sl}(2)$. Last, for all $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{sl}(2)$, the matrix product, $\boldsymbol{AB} \in \mathfrak{sl}(2)$, since $|\boldsymbol{AB}| = |\boldsymbol{A}| \times |\boldsymbol{B}| = 1$. Thus, matrix multiplication is the multiplicative operator for this group, matrix inversion is the inverse operator, and multiplication by the 2×2 identity matrix is the identity operator. The restriction to a unit determinant is a normalization that gives three independent parameters in the group.

The matrix $\boldsymbol{A} \in \mathfrak{sl}(2)$ in (A.19) is associated with the projective group transformation $y(x) = (\alpha x + \beta)/(\gamma x + \delta)$ for all $x \in \mathbb{R}$ such that $|y(x)| < \infty$, a rational map with linear functions in both the numerator and the denominator. The inverse projective group transformation, $x(y) = (\delta y - \beta)/(-\gamma y + \alpha)$, is associated with $\boldsymbol{A}^{-1} \in \mathfrak{sl}(2)$. $\boldsymbol{I}_2$ defines the identity map $y(x) = x$. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{sl}(2)$ define a pair of projective group transformations by $a(x) = (a_{11}x + a_{12})/(a_{21}x + a_{22})$ and $b(x) = (b_{11}x + b_{12})/(b_{21}x + b_{22})$. The matrix product $\boldsymbol{BA}$ is associated with the composition of the projective group transformations,

$$b(a(x)) = \frac{(b_{11}a_{11} + b_{12}a_{21})x + (b_{11}a_{12} + b_{12}a_{22})}{(b_{21}a_{11} + b_{22}a_{21})x + (b_{21}a_{12} + b_{22}a_{22})}. \tag{A.21}$$

This can be verified by computing the individual elements of ***BA*** and rearranging terms in the composite function $b(a(x))$. That is, *multiplication* is the composition of transformations in the group and this is one-to-one and onto with matrix multiplication in $\mathfrak{sl}(2)$. A simple inductive argument implies that any sequence of compositions of projective group transformations is a projective group transformation, thereby completing the set.

A.3 Proof of Proposition 1

Proposition 1: *Let* $\pi : \mathbb{R}_+^n \to \mathbb{R}_+$, $\pi \in \mathcal{C}^\infty$*, be positive-valued, 1° homogeneous, increasing, and concave in* ***p****; let* $\eta : \mathbb{R}_+^n \to \mathbb{R}$, $\eta \in \mathcal{C}^\infty$*, be 0° homogeneous in* ***p****; let* $\alpha, \beta, \gamma, \delta : \mathbb{R}_+^n \to \mathbb{C}$, $\alpha, \beta, \gamma, \delta \in \mathcal{C}^\infty$*, be 0° homogeneous and satisfy the normalizing identity* $\alpha\delta - \beta\gamma \equiv 1$*; and let* $f : \mathbb{R} \to \mathbb{C}$, $f \in \mathcal{C}^\infty$, $f' \neq 0$*. Then the expenditure function for a full rank Gorman or Lewbel system is a special case of:*

$$f\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right) = \frac{\alpha(\boldsymbol{p})w(\eta(\boldsymbol{p}),u) + \beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})w(\eta(\boldsymbol{p}),u) + \delta(\boldsymbol{p})}. \tag{A.22}$$

Proof: We first prove the representation (A.22) for all Gorman systems and then for all Lewbel systems.

Gorman systems

Full rank one can always be written as $e(\boldsymbol{p},u)/\pi(\boldsymbol{p}) = u$, $\pi(\boldsymbol{p})$ 1° homogeneous because adding up and ordinal utility imply that WLOG, $f(x) = x$. Full rank two is only slightly more involved. For the PIGL model, we have

$$v(\boldsymbol{p},m) = [m^\kappa - \beta_1(\boldsymbol{p})]/\beta_2(\boldsymbol{p}), \tag{A.23}$$

with $\beta_1(\boldsymbol{p})$ and $\beta_2(\boldsymbol{p})$ $\kappa°$ homogeneous. Rewrite this in terms of deflated expenditure,

$$\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)^{\kappa}=u+\beta(\boldsymbol{p}), \tag{A.24}$$

with $\pi(\boldsymbol{p})\equiv\beta_2(\boldsymbol{p})^{1/\kappa}$ 1° homogeneous and $\beta(\boldsymbol{p})\equiv\beta_1(\boldsymbol{p})/\beta_2(\boldsymbol{p})$ 0° homogeneous. For the PIGLOG model, we have

$$v(\boldsymbol{p},m)=[\ln m-\beta_1(\boldsymbol{p})]/\beta_2(\boldsymbol{p}), \tag{A.25}$$

where $\beta_1(\boldsymbol{p})=\ln\tilde{\beta}_1(\boldsymbol{p})$, with $\tilde{\beta}_1(\boldsymbol{p})$ 1° homogeneous, $\beta_2(\boldsymbol{p})$ 0° homogeneous. Rewrite this in terms of deflated expenditure,

$$\ln\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)=\alpha(\boldsymbol{p})u, \tag{A.26}$$

where $\pi(\boldsymbol{p})\equiv\tilde{\beta}_1(\boldsymbol{p})$ is 1° homogeneous and $\alpha(\boldsymbol{p})\equiv\beta_2(\boldsymbol{p})$ is 0° homogeneous.

The van Daal and Merkies (1989) and Lewbel (1987, 1990) solution for full rank three Gorman systems with $f(m)\in\{m^{\kappa},\ln m\}$, $\kappa\in\mathbb{R}$, and $\theta(x)\equiv\lambda$, is:

$$\int^{-\beta_3(\boldsymbol{p})/[f(e(\boldsymbol{p},u))-\beta_1(\boldsymbol{p})]}\frac{dw}{(1+\lambda w^2)}=\beta_2(\boldsymbol{p})+u\,. \tag{A.27}$$

We must put each of six cases in the form of a projective group transformation.

Generalized PIGL

For the generalized PIGL and $\lambda>0$, we will use:

$$\int_0^x\frac{ds}{(1+s^2)}=\tan^{-1}(x). \tag{A.28}$$

Let $\lambda=\mu^2>0$ and $s=\mu w$, so that (A.27) becomes

$$\int^{-\beta_3(\boldsymbol{p})/[e(\boldsymbol{p},u)^{\kappa}-\beta_1(\boldsymbol{p})]}\frac{dw}{(1+\lambda w^2)}=\frac{1}{\mu}\tan^{-1}\left\{\frac{-\mu\beta_3(\boldsymbol{p})}{e(\boldsymbol{p},u)^{\kappa}-\beta_1(\boldsymbol{p})}\right\}=\beta_2(\boldsymbol{p})+c(u). \tag{A.29}$$

The functions $\beta_1(\boldsymbol{p})$ and $\beta_3(\boldsymbol{p})$ are $\kappa°$ homogeneous and $\beta_2(\boldsymbol{p})$ is 0° homogeneous. Define $\tilde{\beta}_1(\boldsymbol{p}) \equiv \beta_1(\boldsymbol{p})^{1/\kappa}$ and $\tilde{\beta}_3(\boldsymbol{p}) \equiv \beta_3(\boldsymbol{p})/\beta_1(\boldsymbol{p})$, so that $\tilde{\beta}_1(\boldsymbol{p})$ is 1° homogeneous, while $\tilde{\beta}_3(\boldsymbol{p})$ is 0° homogeneous. Rewrite (A.29) as

$$\frac{-\mu\tilde{\beta}_3(\boldsymbol{p})}{[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})]^{\kappa}-1} = \frac{\tan[\mu c(u)] + \tan[\mu\beta_2(\boldsymbol{p})]}{1-\tan[\mu\beta_2(\boldsymbol{p})]\tan[\mu c(u)]}, \tag{A.30}$$

using the trigonometric rule for finding the tangent of the sum of two angles. Apply the normalization $c(u) = \mu^{-1}\tan^{-1}(u)$ with $\tan^{-1}(0) = 0$, and rearrange terms to yield:

$$\left(\frac{e(\boldsymbol{p},u)}{\tilde{\beta}_1(\boldsymbol{p})}\right)^{\kappa} = \frac{\left\{\tan[\mu\beta_2(\boldsymbol{p})] + \mu\tilde{\beta}_3(\boldsymbol{p})\right\}u + \mu\tilde{\beta}_3(\boldsymbol{p})\tan[\mu\beta_2(\boldsymbol{p})] - 1}{\tan[\mu\beta_2(\boldsymbol{p})]u - 1}. \tag{A.31}$$

We have:

$$\begin{aligned} &-\tan[\mu\beta_2(\boldsymbol{p})] - \mu\tilde{\beta}_3(\boldsymbol{p}) - \left[\mu\tilde{\beta}_3(\boldsymbol{p})\tan(\mu\beta_2(\boldsymbol{p})) - 1\right]\tan(\mu\beta_2(\boldsymbol{p})) \\ &= -\mu\tilde{\beta}_3(\boldsymbol{p})/\cos^2(\mu\beta_2(\boldsymbol{p})). \end{aligned} \tag{A.32}$$

Define $\pi(\boldsymbol{p}) = \tilde{\beta}_1(\boldsymbol{p}) \cdot \left[-\mu\tilde{\beta}_3(\boldsymbol{p})/\cos^2(\mu\beta_2(\boldsymbol{p}))\right]^{1/2\kappa}$;

$\alpha(\boldsymbol{p}) = \left[\sin(\mu\beta_2(\boldsymbol{p})) + \mu\tilde{\beta}_3(\boldsymbol{p})\cos(\mu\beta_2(\boldsymbol{p}))\right]/\sqrt{-\mu\tilde{\beta}_3(\boldsymbol{p})}$;

$\beta(\boldsymbol{p}) = \left[\mu\tilde{\beta}_3(\boldsymbol{p})\sin(\mu\beta_2(\boldsymbol{p})) - \cos(\mu\beta_2(\boldsymbol{p}))\right]/\sqrt{-\mu\tilde{\beta}_3(\boldsymbol{p})}$;

$\gamma(\boldsymbol{p}) = \sin(\mu\beta_2(\boldsymbol{p}))/\sqrt{-\mu\tilde{\beta}_3(\boldsymbol{p})}$; and

$\delta(\boldsymbol{p}) = -\cos(\mu\beta_2(\boldsymbol{p}))/\sqrt{-\mu\tilde{\beta}_3(\boldsymbol{p})}$.

Because $\tilde{\beta}_1(\boldsymbol{p})$ is 1° homogeneous, while $\beta_2(\boldsymbol{p})$ and $\tilde{\beta}_3(\boldsymbol{p})$ are 0° homogeneous, $\pi(\boldsymbol{p})$ is 1° homogeneous, while $\alpha(\boldsymbol{p})$, $\beta(\boldsymbol{p})$, $\gamma(\boldsymbol{p})$, and $\delta(\boldsymbol{p})$ are 0° homogeneous. A direct calculation yields $\alpha(\boldsymbol{p})\delta(\boldsymbol{p}) - \beta(\boldsymbol{p})\gamma(\boldsymbol{p}) \equiv 1$, as required. Now rewrite (A.31) as

$$\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)^{\kappa} = \frac{\alpha(\boldsymbol{p})u + \beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})u + \delta(\boldsymbol{p})}. \tag{A.33}$$

The case where $\lambda = 0$ is more straightforward, since

$$\frac{-\beta_3(\boldsymbol{p})}{e(\boldsymbol{p},u)^{\kappa} - \beta_1(\boldsymbol{p})} = \int^{-\beta_3(\boldsymbol{p})/[e(\boldsymbol{p},u)\kappa - \beta_1(\boldsymbol{p})]} dw = \beta_2(\boldsymbol{p}) + c(u)\,. \qquad \text{(A.34)}$$

Define $\tilde{\beta}_1(\boldsymbol{p})$ and $\tilde{\beta}_3(\boldsymbol{p})$ as before and rearrange terms to obtain:

$$\left(\frac{e(\boldsymbol{p},u)}{\tilde{\beta}_1(\boldsymbol{p})}\right)^{\kappa} = 1 - \tilde{\beta}_3(\boldsymbol{p})\beta_2(\boldsymbol{p}) - \tilde{\beta}_3(\boldsymbol{p})c(u). \qquad \text{(A.35)}$$

The obvious normalization here is $c(u) = -u$, so that

$$\left(\frac{e(\boldsymbol{p},u)}{\tilde{\beta}_1(\boldsymbol{p})}\right)^{\kappa} = \tilde{\beta}_3(\boldsymbol{p})u + 1 - \tilde{\beta}_3(\boldsymbol{p})\beta_2(\boldsymbol{p}). \qquad \text{(A.36)}$$

Define $\pi(\boldsymbol{p}) = \tilde{\beta}_1(\boldsymbol{p}) \cdot \tilde{\beta}_3(\boldsymbol{p})^{1/2\kappa}$;

$\alpha(\boldsymbol{P}) = \sqrt{\tilde{\beta}_3(\boldsymbol{p})}$;

$\beta(\boldsymbol{p}) = \left[1 - \tilde{\beta}_3(\boldsymbol{p})\beta_2(\boldsymbol{p})\right] \Big/ \sqrt{\tilde{\beta}_3(\boldsymbol{p})}$;

$\gamma(\boldsymbol{p}) = 0$; and

$\delta(\boldsymbol{p}) = 1 \Big/ \sqrt{\tilde{\beta}_3(\boldsymbol{p})}$.

We again obtain the group representation (A.33).

Next, let $\lambda = -\mu^2 < 0$ in (A.27), so that

$$\begin{aligned} \int^{-\beta_3(\boldsymbol{p})/[e(\boldsymbol{p},u)^{\kappa} - \beta_1(\boldsymbol{p})]} \frac{dw}{(1+\lambda w^2)} &= \int^{-\beta_3(\boldsymbol{p})/[e(\boldsymbol{p},u)^{\kappa} - \beta_1(\boldsymbol{p})]} \frac{dw}{(1+\mu w)(1-\mu w)} \\ &= \frac{1}{2\mu} \ln\left\{\frac{e(\boldsymbol{p},u)^{\kappa} - \beta_1(\boldsymbol{p}) - \mu\beta_3(\boldsymbol{p})}{e(\boldsymbol{p},u)^{\kappa} - \beta_1(\boldsymbol{p}) + \mu\beta_3(\boldsymbol{p})}\right\} \qquad \text{(A.37)} \\ &= \beta_2(\boldsymbol{p}) + c(u). \end{aligned}$$

Defining $\tilde{\beta}_1(\boldsymbol{p})$ and $\tilde{\beta}_3(\boldsymbol{p})$ as before, this can be rewritten as

$$\frac{\left[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})\right]^{\kappa}-1-\mu\tilde{\beta}_3(\boldsymbol{p})}{\left[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})\right]^{\kappa}-1+\mu\beta_3(\boldsymbol{p})}=\exp\left\{2\mu\left[\beta_2(\boldsymbol{p})+c(u)\right]\right\}. \qquad \text{(A.38)}$$

In this case, the obvious normalization is $c(u)=\ln u/2\mu$. Rearranging gives

$$\left(\frac{e(\boldsymbol{p},u)}{\tilde{\beta}_1(\boldsymbol{p})}\right)^{\kappa}=\frac{\left[1-\mu\beta_3(\boldsymbol{p})\right]\mathrm{e}^{2\mu\beta_2(\boldsymbol{p})}\cdot u-\left[1+\mu\tilde{\beta}_3(\boldsymbol{p})\right]}{\mathrm{e}^{2\mu\beta_2(\boldsymbol{p})}\cdot u-1}. \qquad \text{(A.39)}$$

We have:

$$-\left[1-\mu\beta_3(\boldsymbol{p})\right]\mathrm{e}^{2\mu\beta_2(\boldsymbol{p})}+\left[1+\mu\tilde{\beta}_3(\boldsymbol{p})\right]\mathrm{e}^{2\mu\beta_2(\boldsymbol{p})}=2\mu\tilde{\beta}_3(\boldsymbol{p})\mathrm{e}^{2\mu\beta_2(\boldsymbol{p})}. \qquad \text{(A.40)}$$

Define $\pi(\boldsymbol{p})=\tilde{\beta}_1(\boldsymbol{p})\cdot\left[2\mu\tilde{\beta}_3(\boldsymbol{p})\mathrm{e}^{2\mu\tilde{\beta}_3(\boldsymbol{p})}\right]^{1/2\kappa}$;

$\alpha(\boldsymbol{P})=\left[1-\mu\beta_3(\boldsymbol{p})\right]\mathrm{e}^{\mu\beta_2(\boldsymbol{p})}\Big/\sqrt{2\mu\tilde{\beta}_3(\boldsymbol{p})}$;

$\beta(\boldsymbol{p})=-\left[1+\mu\tilde{\beta}_3(\boldsymbol{p})\right]\Big/\left[\sqrt{2\mu\tilde{\beta}_3(\boldsymbol{p})}\mathrm{e}^{\mu\tilde{\beta}_3(\boldsymbol{p})}\right]$;

$\gamma(\boldsymbol{p})=\mathrm{e}^{\mu\beta_2(\boldsymbol{p})}\Big/\sqrt{2\mu\tilde{\beta}_3(\boldsymbol{p})}$; and

$\delta(\boldsymbol{p})=-1\Big/\left[\sqrt{2\mu\tilde{\beta}_3(\boldsymbol{p})}\mathrm{e}^{\mu\tilde{\beta}_3(\boldsymbol{p})}\right]$.

Once again, we obtain the group representation (A.33).

Generalized PIGLOG

The same three cases apply here as for the generalized PIGL case, except that $\ln m$ replaces m^{κ} everywhere, $\beta_1(\boldsymbol{p})=\ln\tilde{\beta}_1(\boldsymbol{P})$ for some 1° homogeneous function, $\tilde{\beta}_1(\boldsymbol{p})$, while both $\beta_2(\boldsymbol{p})$ and $\beta_3(\boldsymbol{p})$ are 0° homogeneous. When $\lambda>0$, (A.29) becomes

$$\int^{-\beta_3(\boldsymbol{p})/\ln[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})]}\frac{dw}{(1+\lambda w^2)}=\frac{1}{\mu}\tan^{-1}\left\{\frac{-\mu\beta_3(\boldsymbol{p})}{\ln[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})]}\right\}=\beta_2(\boldsymbol{p})+c(u). \qquad \text{(A.41)}$$

The same steps as for the generalized PIGL case lead to

$$\ln\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)=\frac{\alpha(\boldsymbol{p})u+\beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})u+\delta(\boldsymbol{p})}, \tag{A.42}$$

with the definitions below (A.32), but with $\pi(\boldsymbol{p})\equiv\tilde{\beta}_1(\boldsymbol{p})$ and $\beta_3(\boldsymbol{p})$ replacing $\tilde{\beta}_3(\boldsymbol{p})$. Similarly, if $\lambda=0$, we obtain (A.42) with the definitions below (A.36), with $\pi(\boldsymbol{p})\equiv\tilde{\beta}_1(\boldsymbol{p})$ and $\beta_3(\boldsymbol{p})$ replacing $\tilde{\beta}_3(\boldsymbol{p})$, while if $\lambda<0$, we obtain (A.42) with the definitions below (A.40), with $\pi(\boldsymbol{p})\equiv\tilde{\beta}_1(\boldsymbol{p})$ and $\beta_3(\boldsymbol{p})$ replacing $\tilde{\beta}_3(\boldsymbol{p})$ everywhere.

For the generalized PIGL and PIGLOG cases with $\theta'(\beta_3(\boldsymbol{p}))\neq 0$, write

$$w(\beta_3(\boldsymbol{p}),u)=\frac{f(e(\boldsymbol{p},u))-\beta_1(\boldsymbol{p})}{\beta_2(\boldsymbol{p})}. \tag{A.43}$$

Then the same steps as above give the group representations.

Trigonometric

From Lewbel (1988, 1990) the indirect utility function is

$$v(\boldsymbol{p},m)=\beta_2(\boldsymbol{p})+\frac{\beta_3(\boldsymbol{p})\cos\left[\tau\ln\left(m/\beta_1(\boldsymbol{p})\right)\right]}{\left[1-\sin\left[\tau\ln\left(m/\beta_1(\boldsymbol{p})\right)\right]\right]}. \tag{A.44}$$

Apply the definitions of and rules for calculating sums and differences of sine and cosine functions (e.g., Abramowitz and Stegun 1972, pp.71-74) to rewrite (A.44) as

$$v(\boldsymbol{p},m)=\frac{\left[\beta_3(\boldsymbol{p})-\iota\beta_2(\boldsymbol{p})\right]\times\left[m/\beta_1(\boldsymbol{p})\right]^{\iota\tau}+\beta_2(\boldsymbol{p})-\iota\beta_3(\boldsymbol{p})}{1-\iota\left[m/\beta_1(\boldsymbol{p})\right]^{\iota\tau}}. \tag{A.45}$$

As before, to find the group representation, we need the appropriate transformation of deflated income. Setting $v(\boldsymbol{p},m)=u$ and $m=e(\boldsymbol{p},u)$ and inverting (A.45) yields:

$$\left(\frac{e(\boldsymbol{p},u)}{\beta_1(\boldsymbol{p})}\right)^{\iota\tau}=\frac{u-\left[\beta_2(\boldsymbol{p})-\iota\cdot\beta_3(\boldsymbol{p})\right]}{\iota\cdot u-\iota\cdot\left[\beta_2(\boldsymbol{p})+\iota\cdot\beta_3(\boldsymbol{p})\right]}. \tag{A.46}$$

We have

$$-\iota\cdot\left[\beta_2(\boldsymbol{p})+\iota\cdot\beta_3(\boldsymbol{p})\right]+\iota\cdot\left[\beta_2(\boldsymbol{p})-\iota\cdot\beta_3(\boldsymbol{p})\right]=2\beta_3(\boldsymbol{p}). \tag{A.47}$$

Define $\pi(\boldsymbol{p})=\beta_1(\boldsymbol{p})\cdot\left[2\beta_3(\boldsymbol{p})\right]^{1/(2\iota\tau)}$;

$\alpha(\boldsymbol{p})=1\big/\sqrt{2\beta_3(\boldsymbol{p})}$;

$\beta(\boldsymbol{p})=-\left[\beta_2(\boldsymbol{p})-\iota\cdot\beta_3(\boldsymbol{p})\right]\big/\sqrt{2\beta_3(\boldsymbol{p})}$;

$\gamma(\boldsymbol{p})=\iota\big/\sqrt{2\beta_3(\boldsymbol{p})}$; and

$\delta(\boldsymbol{p})=-\iota\cdot\left[\beta_2(\boldsymbol{p})+\iota\cdot\beta_3(\boldsymbol{p})\right]\big/\sqrt{2\beta_3(\boldsymbol{p})}$.

This yields,

$$\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)^{\iota\tau}=\frac{\alpha(\boldsymbol{p})u+\beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})u+\delta(\boldsymbol{p})}. \tag{A.48}$$

Lewbel systems

Recall equation (12) in the text,

$$\frac{\partial\left(e(\boldsymbol{p},u)/\pi(\boldsymbol{p})\right)}{\partial\boldsymbol{p}}=\sum_{k=2}^{K}\tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p})h_k\left(m/\pi(\boldsymbol{p})\right). \tag{A.49}$$

If $K\geq 2$, linear independence of $\{h_2(x),\ldots,h_K(x)\}$ implies that at least one cannot vanish. WLOG, let it be $h_2(x)$ and define

$$y(\boldsymbol{p},u)=f\left(e(\boldsymbol{p},u)/\pi(\boldsymbol{p})\right)=\int^{e(\boldsymbol{p},u)/\pi(\boldsymbol{p})}\frac{dx}{h_2(x)}. \tag{A.50}$$

By Leibnitz' rule, we have

$$\begin{aligned}\frac{\partial y(\boldsymbol{p},u)}{\partial\boldsymbol{p}}&=\frac{1}{h_2(e(\boldsymbol{p},u)/\pi(\boldsymbol{p}))}\left[\frac{\boldsymbol{q}}{\pi(\boldsymbol{p})}-\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})^2}\frac{\partial\pi(\boldsymbol{p})}{\partial\boldsymbol{p}}\right]\\&=\tilde{\boldsymbol{\alpha}}_2(\boldsymbol{p})+\sum_{k=3}^{K}\tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p})\frac{h_k\left(e(\boldsymbol{p},u/\pi(\boldsymbol{p})\right)}{h_2(e(\boldsymbol{p},u)/\pi(\boldsymbol{p}))}\\&\equiv\tilde{\boldsymbol{\alpha}}_2(\boldsymbol{p})+\sum_{k=3}^{K}\tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p})\tilde{h}_k\left(e(\boldsymbol{p},u/\pi(\boldsymbol{p})\right).\end{aligned} \tag{A.51}$$

These steps reduce the problem to one in which the first income term on the right-hand side is one and maintains the Gorman structure. Since $h_2(x) \neq 0$, $f^{-1}(y)$ exists, so that

$$\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \tilde{\boldsymbol{\alpha}}_2(\boldsymbol{p}) + \sum_{k=3}^{K} \tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p})\hat{h}_k\left(y(\boldsymbol{p},u)\right), \tag{A.52}$$

where $\hat{h}_k\left(y(\boldsymbol{p},u)\right) = \tilde{h}_k\left(f^{-1}\left(y(\boldsymbol{p},u)\right)\right)$, $\hat{h}_k : \mathbb{R} \to \mathbb{R}$, $\hat{h}_k \in \mathcal{C}^{\infty}$, $k = 3,\cdots,K$.

We know from Lewbel (1989a) that $K \leq 4$. Hence, we must find all of the solutions to (A.52) for K=2,3,4. To simplify notation, drop the ~'s and ^'s to rewrite (A.52) as

$$\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2(\boldsymbol{p}) + \sum\nolimits_{k=3}^{K} \boldsymbol{\alpha}_k(\boldsymbol{p})h_k\left(y(\boldsymbol{p},u)\right). \tag{A.53}$$

K=2:

$$\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2(\boldsymbol{p}). \tag{A.54}$$

This implies

$$\frac{\partial^2 y}{\partial \boldsymbol{p} \partial \boldsymbol{p}^{\top}} = \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{p}^{\top}}, \tag{A.55}$$

so that $\partial \boldsymbol{\alpha}_2 / \partial \boldsymbol{p}^{\top}$ must be symmetric. This is necessary and sufficient for the existence of a function, $\beta : \mathbb{R}_+^n \to \mathbb{R}$, $\beta \in \mathcal{C}^{\infty}$, such that $\partial \beta(\boldsymbol{p}) / \partial \boldsymbol{p} = \boldsymbol{\alpha}_2(\boldsymbol{p})$. Integrating (A.54) yields

$$y(\boldsymbol{p},u) = \beta(\boldsymbol{p}) + u, \tag{A.56}$$

with an obvious normalization for the constant of integration.

K=3:

$$\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2(\boldsymbol{p}) + \boldsymbol{\alpha}_3(\boldsymbol{p})h_3\left(y(\boldsymbol{p},u)\right). \tag{A.57}$$

This implies

$$\frac{\partial^2 y}{\partial \boldsymbol{p}\partial \boldsymbol{p}^\top} = \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{p}^\top} + \frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{p}^\top} h_3 + \boldsymbol{\alpha}_3\boldsymbol{\alpha}_2^\top h_3' + \boldsymbol{\alpha}_3\boldsymbol{\alpha}_3^\top h_3 h_3'$$
$$= \frac{\partial \boldsymbol{\alpha}_2^\top}{\partial \boldsymbol{p}} + \frac{\partial \boldsymbol{\alpha}_3^\top}{\partial \boldsymbol{p}} h_3 + \boldsymbol{\alpha}_2\boldsymbol{\alpha}_3^\top h_3' + \boldsymbol{\alpha}_3\boldsymbol{\alpha}_3^\top h_3 h_3'. \tag{A.58}$$

Subtracting the second line from the first implies,

$$\left(\boldsymbol{\alpha}_3\boldsymbol{\alpha}_2^\top - \boldsymbol{\alpha}_2\boldsymbol{\alpha}_3^\top\right) h_3' = \left(\frac{\partial \boldsymbol{\alpha}_2^\top}{\partial \boldsymbol{p}} - \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{p}^\top}\right) + \left(\frac{\partial \boldsymbol{\alpha}_3^\top}{\partial \boldsymbol{p}} - \frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{p}^\top}\right) h_3. \tag{A.59}$$

Since $\{\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$ are linearly independent, $\boldsymbol{\alpha}_3 \neq c\boldsymbol{\alpha}_2$ for any $c \in \mathbb{R}$. Hence, $\boldsymbol{\alpha}_3\boldsymbol{\alpha}_2^\top$ is not symmetric. Since $\{1, h_3(y)\}$ are linearly independent, $h_3' \neq 0$. Premultiply (A.59) by $\boldsymbol{\alpha}_3^\top$, postmultiply by $\boldsymbol{\alpha}_2$, and divide by $\boldsymbol{\alpha}_3^\top\boldsymbol{\alpha}_3\boldsymbol{\alpha}_2^\top\boldsymbol{\alpha}_2 - (\boldsymbol{\alpha}_3^\top\boldsymbol{\alpha}_2)^2 \neq 0$ to obtain

$$h_3'(y) = c_1 + c_2 h_3(y), \tag{A.60}$$

with c_1 and c_2 absolute constants since $h_3(y)$ and $h_3'(y)$ are independent of $\boldsymbol{p}$.

If $c_2 \neq 0$, then applying the integrating factor $\mathrm{e}^{-c_2 y}$ implies that the general solution to this linear, first-order, ordinary differential equation has the form

$$h_3(y) = -(c_1/c_2) + c_3\mathrm{e}^{c_2 y}, \tag{A.61}$$

where c_3 is a constant of integration. Plugging this into (A.59)

$$\left(\boldsymbol{\alpha}_3\boldsymbol{\alpha}_2^\top - \boldsymbol{\alpha}_2\boldsymbol{\alpha}_3^\top\right) c_2 c_3 \mathrm{e}^{c_2 y} = \left(\frac{\partial \boldsymbol{\alpha}_2^\top}{\partial \boldsymbol{p}} - \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{p}^\top}\right) + \left(\frac{\partial \boldsymbol{\alpha}_3^\top}{\partial \boldsymbol{p}} - \frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{p}^\top}\right)\left[-(c_1/c_2) + c_3\mathrm{e}^{c_2 y}\right]. \tag{A.62}$$

This implies $c_3=0$, contradicting the linear independence of $\{1, h_3(y)\}$.

Therefore, it must be that $c_2=0$, and the solution to (A.60) is $h_3(y) = c_1 y$. This reduces the demand system to

$$\frac{\partial y(\boldsymbol{p}, u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2(\boldsymbol{p}) + \boldsymbol{\alpha}_3(\boldsymbol{p}) y(\boldsymbol{p}, u). \tag{A.63}$$

Symmetry now reduces to

$$\frac{\partial^2 y}{\partial \boldsymbol{p}\partial \boldsymbol{p}^\top} = \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{p}^\top} + \boldsymbol{\alpha}_3\boldsymbol{\alpha}_2^\top + \left(\frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{p}^\top} + \boldsymbol{\alpha}_3\boldsymbol{\alpha}_3^\top\right) y = \frac{\partial \boldsymbol{\alpha}_2^\top}{\partial \boldsymbol{p}} + \boldsymbol{\alpha}_2\boldsymbol{\alpha}_3^\top + \left(\frac{\partial \boldsymbol{\alpha}_3^\top}{\partial \boldsymbol{p}} + \boldsymbol{\alpha}_3\boldsymbol{\alpha}_3^\top\right) y. \tag{A.64}$$

Equating like powers in y, $\partial \boldsymbol{\alpha}_3 \big/ \partial \boldsymbol{p}^\top$ is symmetric. Hence, a 0º homogeneous function, $\delta : \mathbb{R}_+^n \to \mathbb{R}$, $\delta \in \mathcal{C}^\infty$, exists such that $\partial \delta(\boldsymbol{p}) / \partial \boldsymbol{p} = \boldsymbol{\alpha}_3(\boldsymbol{p})$ and $\partial \boldsymbol{\alpha}_2 \big/ \partial \boldsymbol{p}^\top - \boldsymbol{\alpha}_2\, \partial \delta \big/ \partial \boldsymbol{p}^\top$ is symmetric. Applying the integrating factor $\mathrm{e}^{-\delta}$ to

$$\frac{\partial}{\partial \boldsymbol{p}} y(\boldsymbol{p},u)\mathrm{e}^{-\delta(\boldsymbol{p})} = \left[\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} - y(\boldsymbol{p},u)\frac{\partial \delta(\boldsymbol{p})}{\partial \boldsymbol{p}}\right]\mathrm{e}^{-\delta(\boldsymbol{p})}, \tag{A.65}$$

and

$$\frac{\partial}{\partial \boldsymbol{p}^\top}\left[\boldsymbol{\alpha}_2(\boldsymbol{p})\mathrm{e}^{-\delta(\boldsymbol{p})}\right] = \left[\frac{\partial \boldsymbol{\alpha}_2(\boldsymbol{p})}{\partial \boldsymbol{p}^\top} - \boldsymbol{\alpha}_2(\boldsymbol{p})\frac{\partial \delta(\boldsymbol{p})}{\partial \boldsymbol{p}^\top}\right]\mathrm{e}^{-\delta(\boldsymbol{p})}, \tag{A.66}$$

implies that there is a 0º homogeneous function, $\gamma : \mathcal{P} \to \mathbb{R}$, $\gamma \in \mathcal{C}^\infty$, such that

$$y(\boldsymbol{p},u) = u\mathrm{e}^{\delta(\boldsymbol{p})} + \gamma(\boldsymbol{p})\mathrm{e}^{\delta(\boldsymbol{p})} \equiv \alpha(\boldsymbol{p})u + \beta(\boldsymbol{p}), \tag{A.67}$$

with u as the constant of integration.

K=4:
$$\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2(\boldsymbol{p}) + \boldsymbol{\alpha}_3(\boldsymbol{p})h_3\left(y(\boldsymbol{p},u)\right) + \boldsymbol{\alpha}_4(\boldsymbol{p})h_4\left(y(\boldsymbol{p},u)\right). \tag{A.68}$$

We have

$$\begin{aligned}\frac{\partial^2 y}{\partial p_i \partial p_j} &= \frac{\partial \alpha_{i2}}{\partial p_j} + \sum_{k=3}^{4}\frac{\partial \alpha_{ik}}{\partial p_j}h_k + \sum_{k=3}^{4}\alpha_{ik}h_k'\left(\alpha_{j2} + \sum_{\ell=3}^{4}\alpha_{j\ell}h_\ell\right)\\ &= \frac{\partial \alpha_{j2}}{\partial p_i} + \sum_{k=3}^{4}\frac{\partial \alpha_{jk}}{\partial p_i}h_k + \sum_{k=3}^{4}\alpha_{jk}h_k'\left(\alpha_{i2} + \sum_{\ell=3}^{4}\alpha_{i\ell}h_\ell\right) = \frac{\partial^2 y}{\partial p_j \partial p_i},\ \forall\ i \neq j.\end{aligned} \tag{A.69}$$

Rewrite this in terms of ½n(n–1) vanishing differences,

$$0=\frac{\partial\alpha_{i2}}{\partial p_j}-\frac{\partial\alpha_{j2}}{\partial p_i}+\left(\frac{\partial\alpha_{i3}}{\partial p_j}-\frac{\partial\alpha_{j3}}{\partial p_i}\right)h_3+\left(\frac{\partial\alpha_{i4}}{\partial p_j}-\frac{\partial\alpha_{j4}}{\partial p_i}\right)h_4$$
$$+\left(\alpha_{i3}\alpha_{j2}-\alpha_{i2}\alpha_{j3}\right)h_3'+\left(\alpha_{i4}\alpha_{j2}-\alpha_{i2}\alpha_{j4}\right)h_4' \tag{A.70}$$
$$+\sum_{k=3}^{4}\sum_{\ell=3}^{4}\alpha_{ik}\alpha_{j\ell}\left(h_k'h_\ell-h_kh_\ell'\right),\ \forall\ j<i=2,\cdots,n.$$

If $k=\ell$ in the double sum, then $\alpha_{ik}\alpha_{jk}$ is multiplied $h_k'h_k-h_kh_k'=0$, while if $k\neq\ell$, then $h_k'h_\ell-h_kh_\ell'$ is multiplied once by $\alpha_{ik}\alpha_{j\ell}$ and once by $-\alpha_{i\ell}\alpha_{jk}$. Rewrite (A.70) as

$$0=\frac{\partial\alpha_{i2}}{\partial p_j}-\frac{\partial\alpha_{j2}}{\partial p_i}+\left(\frac{\partial\alpha_{i3}}{\partial p_j}-\frac{\partial\alpha_{j3}}{\partial p_i}\right)h_3+\left(\frac{\partial\alpha_{i4}}{\partial p_j}-\frac{\partial\alpha_{j4}}{\partial p_i}\right)h_4$$
$$+\left(\alpha_{i3}\alpha_{j2}-\alpha_{i2}\alpha_{j3}\right)h_3'+\left(\alpha_{i4}\alpha_{j2}-\alpha_{i2}\alpha_{j4}\right)h_4' \tag{A.71}$$
$$+\left(\alpha_{i4}\alpha_{j3}-\alpha_{i3}\alpha_{j4}\right)\left(h_3'h_4-h_3h_4'\right),\ \forall\ j<i=2,\cdots,n.$$

Define

$$\boldsymbol{B}=\begin{bmatrix}\alpha_{23}\alpha_{12}-\alpha_{22}\alpha_{13} & \alpha_{24}\alpha_{12}-\alpha_{22}\alpha_{14} & \alpha_{24}\alpha_{13}-\alpha_{23}\alpha_{14}\\ \alpha_{33}\alpha_{12}-\alpha_{32}\alpha_{13} & \alpha_{34}\alpha_{12}-\alpha_{32}\alpha_{14} & \alpha_{34}\alpha_{13}-\alpha_{13}\alpha_{34}\\ \vdots & \vdots & \vdots\\ \alpha_{n,3}\alpha_{n-1,2}-\alpha_{n,2}\alpha_{n-1,3} & \alpha_{n,4}\alpha_{n-1,2}-\alpha_{n,2}\alpha_{n-13} & \alpha_{n,4}\alpha_{n-1,3}-\alpha_{n-1,3}\alpha_{n,4}\end{bmatrix},$$

$$\boldsymbol{C}=\begin{bmatrix}\frac{\partial\alpha_{22}}{\partial p_2}-\frac{\partial\alpha_{12}}{\partial p_1} & \frac{\partial\alpha_{23}}{\partial p_2}-\frac{\partial\alpha_{13}}{\partial p_1} & \frac{\partial\alpha_{24}}{\partial p_2}-\frac{\partial\alpha_{14}}{\partial p_1}\\ \vdots & \vdots & \vdots\\ \frac{\partial\alpha_{n,2}}{\partial p_{n-1}}-\frac{\partial\alpha_{n-1,2}}{\partial p_n} & \frac{\partial\alpha_{n,3}}{\partial p_{n-1}}-\frac{\partial\alpha_{n-1,3}}{\partial p_n} & \frac{\partial\alpha_{n,4}}{\partial p_{n-1}}-\frac{\partial\alpha_{n-1,4}}{\partial p_n}\end{bmatrix},$$

$\boldsymbol{h}=\begin{bmatrix}1 & h_3 & h_4\end{bmatrix}^{\mathsf{T}}$, and

$\tilde{\boldsymbol{h}}=\begin{bmatrix}h_3' & h_4' & h_3'h_4-h_3h_4'\end{bmatrix}^{\mathsf{T}}$.

$\boldsymbol{B}$ is $½n(n-1)\times 3$, $\boldsymbol{C}$ is $½n(n-1)\times 3$, $\boldsymbol{h}$ is 3×1, and $\tilde{\boldsymbol{h}}$ is 3×1. Full rank requires $n\geq 3$. This gives the symmetry equations as $\boldsymbol{B}\tilde{\boldsymbol{h}}=\boldsymbol{C}\boldsymbol{h}$. Premultiply both sides by $\boldsymbol{B}^{\mathsf{T}}$ to obtain

$\boldsymbol{B}^{\top}\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{B}^{\top}\boldsymbol{C}\boldsymbol{h}$. The symmetric, positive definite matrix $\boldsymbol{B}^{\top}\boldsymbol{B}$ is 3×3 and has rank 3, so that the solution for $\tilde{\boldsymbol{h}}$ in terms of $\boldsymbol{h}$ is

$$\tilde{\boldsymbol{h}} = (\boldsymbol{B}^{\top}\boldsymbol{B})^{-1}\boldsymbol{B}^{\top}\boldsymbol{C}\boldsymbol{h} \equiv \boldsymbol{D}\boldsymbol{h}. \tag{A.72}$$

The vectors $\tilde{\boldsymbol{h}}$ and $\boldsymbol{h}$ depend only on y and not on $\boldsymbol{p}$. The matrix $\boldsymbol{D}$ depends only on $\boldsymbol{p}$ and not on y. It follows that all of the elements of $\boldsymbol{D}$ are absolute constants.

The implications of symmetry on the income functions therefore can be written as

$$\begin{aligned} h_3'(y) &= d_{11} + d_{12}h_3(y) + d_{13}h_4(y), \\ h_4'(y) &= d_{21} + d_{22}h_3(y) + d_{23}h_4(y), \\ h_3(y)h_4'(y) - h_3'(y)h_4(y) &= d_{31} + d_{32}h_3(y) + d_{34}h_4(y), \end{aligned} \tag{A.73}$$

where the $\{d_{ij}\}$ are constants and cannot all be zero in any given equation. The first two equations form a complete system of linear, ordinary differential equations with constant coefficients. This system is constrained by the third equation, which restricts the values that the $\{d_{ij}\}$ can assume in an integrable system.

Differentiate the first differential equation and substitute out $h_4'(y)$ and then $h_4(y)$,

$$\begin{aligned} h_3''(y) &= d_{12}h_3'(y) + d_{13}h_4'(y) \\ &= d_{12}h_3'(y) + d_{13}\left[d_{21} + d_{22}h_3(y) + d_{23}h_4(y)\right] \\ &= d_{13}d_{21} + d_{12}h_3'(y) + d_{13}d_{22}h_3(y) + d_{23}\left[h_3'(y) - d_{11} - d_{12}h_3(y)\right] \\ &= d_{13}d_{21} - d_{22}d_{11} + (d_{11} + d_{22})h_3'(y) + (d_{13}d_{22} - d_{23}d_{12})h_3(y). \end{aligned} \tag{A.74}$$

The homogeneous differential equation is

$$h_3''(y) - (d_{11} + d_{22})h_3'(y) - (d_{13}d_{22} - d_{23}d_{12})h_3(y) = 0, \tag{A.75}$$

with characteristic equation

$$\lambda^2 - (d_{11} + d_{22})\lambda - (d_{13}d_{22} - d_{23}d_{12}) = 0, \tag{A.76}$$

and characteristic roots

$$\lambda = ½\left[d_{11} + d_{12} \pm \sqrt{(d_{11} + d_{12})^2 + 4(d_{13}d_{22} - d_{23}d_{12})}\right]. \tag{A.77}$$

If $\lambda = 0$ is the only root, the complete solution has the form

$$\begin{aligned} h_3(y) &= a_1 + b_1 y + c_1 y^2, \\ h_4(y) &= a_2 + b_2 y + c_2 y^2. \end{aligned} \tag{A.78}$$

We prove that this is the only possibility. With distinct roots, the complete solution to the ordinary differential equations is

$$\begin{aligned} h_3(y) &= a_1 + b_1 e^{\lambda_1 y} + c_1 e^{\lambda_2 y}, \\ h_4(y) &= a_2 + b_2 e^{\lambda_1 y} + c_2 e^{\lambda_2 y}. \end{aligned} \tag{A.79}$$

The first income function is unity, hence, set $h_3(y) = e^{\lambda_1 y}$ and $h_4(y) = e^{\lambda_2 y}$, WLOG. The equation for $h_3 h_4' - h_3' h_4$ then is

$$(\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2) y} = d_{31} + d_{32} e^{\lambda_1 y} + d_{33} e^{\lambda_2 y}, \tag{A.80}$$

with $\lambda_2 - \lambda_1 = \sqrt{(d_{11} + d_{12})^2 + 4(d_{13} d_{22} - d_{23} d_{12})} \neq 0$ and $\lambda_1 + \lambda_2 = d_{11} + d_{12} \neq \lambda_1 \neq \lambda_2$, a contradiction for all $(\lambda_1, \lambda_2) \neq (0,0)$.

Hence, the roots must be equal, $\lambda = ½(d_{11} + d_{12})$. The complete solution then is

$$\begin{aligned} h_3(y) &= a_1 + b_1 e^{\lambda y} + c_1 y e^{\lambda y}, \\ h_4(y) &= a_2 + b_2 e^{\lambda y} + c_2 y e^{\lambda y}. \end{aligned} \tag{A.81}$$

Again WLOG, set $h_3(y) = e^{\lambda y}$ and $h_4(y) = y e^{\lambda y}$. The equation for $h_3 h_4' - h_3' h_4$ is

$$e^{2\lambda y} = d_{31} + d_{32} e^{\lambda y} + d_{33} y e^{\lambda y}, \tag{A.82}$$

a contradiction for all $\lambda \neq 0$. Hence, only a repeated vanishing root is possible and

$$\frac{\partial y}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3 y + \boldsymbol{\alpha}_4 y^2. \tag{A.83}$$

This system has the same form and solutions as a nominal income full rank QES. ■

A.4 Representation Results for Directly Specified Indirect Utility Functions

Translog

$$v(\boldsymbol{p},m)=\boldsymbol{\alpha}^{\top}\ln(\boldsymbol{p}/m)+½\ln(\boldsymbol{p}/m)^{\top}\boldsymbol{B}\ln(\boldsymbol{p}/m)\,. \tag{A.84}$$

If the translog indirect utility function is not exactly aggregable, it is common practice to apply the normalization $\boldsymbol{i}^{\top}\boldsymbol{B}\boldsymbol{i}=1$ for identification, and we can rewrite (A.84) as

$$(\ln e(\boldsymbol{p},u))^{2}+2(\boldsymbol{i}^{\top}\boldsymbol{\alpha}-\boldsymbol{i}^{\top}\boldsymbol{B}\ln\boldsymbol{p})\ln e(\boldsymbol{p},u)+2\boldsymbol{\alpha}^{\top}\ln\boldsymbol{p}+\ln\boldsymbol{p}^{\top}\boldsymbol{B}\ln\boldsymbol{p}-2u=0\,. \tag{A.85}$$

This can be solved for the log-expenditure function as

$$\ln e(\boldsymbol{p},u)=\boldsymbol{i}^{\top}\boldsymbol{B}\ln\boldsymbol{p}-\boldsymbol{i}^{\top}\boldsymbol{\alpha}\pm\sqrt{(\boldsymbol{i}^{\top}\boldsymbol{B}\ln\boldsymbol{p}-\boldsymbol{i}^{\top}\boldsymbol{\alpha})^{2}+2(u-\boldsymbol{\alpha}^{\top}\ln\boldsymbol{p})-\ln\boldsymbol{p}^{\top}\boldsymbol{B}\ln\boldsymbol{p}}\,. \tag{A.86}$$

Since $\pi(\boldsymbol{p})\equiv \mathrm{e}^{-\Sigma_{i=1}^{n}\alpha_i}\prod_{i=1}^{n}p_i^{\Sigma_{j=1}^{n}b_{ij}}$ is 1° homogeneous, subtract $\ln\pi(\boldsymbol{p})$ from both sides of (A.86) and square the result to obtain,

$$½\left[\ln\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)\right]^{2}+\left[\boldsymbol{\alpha}^{\top}\ln\boldsymbol{p}+½\ln\boldsymbol{p}^{\top}\boldsymbol{B}\ln\boldsymbol{p}-½(\boldsymbol{i}^{\top}\boldsymbol{B}\ln\boldsymbol{p}-\boldsymbol{i}^{\top}\boldsymbol{\alpha})^{2}\right]=u\,. \tag{A.87}$$

This is a full rank two system that does not satisfy proposition 2, and is well-known not to be globally regular.

MAIDS

The indirect utility function for the MAIDS (Cooper and McLaren 1992) is

$$v(\boldsymbol{p},m)=\ln\left(\frac{m}{\pi_1(\boldsymbol{p})}\right)\Big/\left(\frac{m}{\pi_2(\boldsymbol{p})}\right)^{\eta}, \tag{A.88}$$

where $\ln\pi_1(\boldsymbol{p})=\kappa+\boldsymbol{\alpha}^{\top}\ln\boldsymbol{p}+½(\ln\boldsymbol{p})^{\top}\boldsymbol{C}\ln\boldsymbol{p}$, $\ln\pi_2(\boldsymbol{p})=(\boldsymbol{\beta}^{\top}\ln\boldsymbol{p})/\eta$, $\boldsymbol{i}^{\top}\boldsymbol{\alpha}=1$, $\boldsymbol{C}=\boldsymbol{C}^{\top}$, $\Sigma_{j=1}^{n}c_{ij}=0,\ i=1,\cdots,n$, and $\boldsymbol{i}^{\top}\boldsymbol{\beta}=\eta\in[0,1]$. Taking the log of both sides yields

$$\ln\left[\ln\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)\right]-\eta\ln\left(\frac{e(\boldsymbol{p},u)}{\pi_2(\boldsymbol{p})}\right)=u\,, \tag{A.89}$$

with $\ln v(\boldsymbol{p},m)=u$. This is a full rank two system that does not satisfy proposition 2.

GEF

The indirect utility function for the GEF (Cooper and McLaren 1996) is

$$v(\boldsymbol{p},m)=\left[\frac{\left(m/\pi_1(\boldsymbol{p})\right)^{\mu}-1}{\mu}\right]\left(\frac{m}{\pi_2(\boldsymbol{p})}\right)^{\eta}, \tag{A.90}$$

where $\pi_1,\pi_2:\mathscr{P}\rightarrow\mathbb{R}_{++}$ are 1° homogeneous, $\mu\geq-1$, and $\eta\in[0,1]$.[15] Taking the log of both sides then gives

$$\ln\left[\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)^{\mu}-1\right]+\eta\ln\left(\frac{e(\boldsymbol{p},u)}{\pi_2(\boldsymbol{p})}\right)=u\,, \tag{A.91}$$

with $\ln\left(\mu v(\boldsymbol{p},m)\right)=u$. This is a full rank two system that does not satisfy proposition 2.

Fractional Demand Systems

The indirect utility functions for fractional demand systems presented in Lewbel (1987b) include homothetic, PIGL, PIGLOG, and translog demand systems – which have already been analyzed – plus three new ones:

1. LOG2 $v(\boldsymbol{p},m)=[A(\boldsymbol{p})+\ln m-1]m\mathrm{e}^{-B(\boldsymbol{p})}$;
2. EXP $v(\boldsymbol{p},m)=\left[m+A(\boldsymbol{p})m^{1+\kappa}\right]e^{-B(\boldsymbol{p})}$; and
3. TAN $v(\boldsymbol{p},m)=A(\boldsymbol{p})\sin(\tau\ln m)+B(\boldsymbol{p})\cos(\tau\ln m)$.

To put each of the new cases in the form of the class of rational demand systems, we must impose the associated homogeneity conditions on the price functions.

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[15] Cooper and McLaren (1996) include a parameter, $\kappa\in[0,1]$ as a multiplicative scalar on $\pi_1(\boldsymbol{p})$ in the definition of the GEF. This parameter can not be identifed in empirical applications and has no effect on the structure of the indirect utility function. Hence, we omit it for compactness.

For the LOG2 indirect utility function, 0° homogeneity requires $A(\boldsymbol{p}) = -\ln \pi_1(\boldsymbol{p})$ and $\mathrm{e}^{B(\boldsymbol{p})} = \pi_2(\boldsymbol{p})$, where $\pi_1(\boldsymbol{p})$ and $\pi_2(\boldsymbol{p})$ are 1° homogeneous. This model can then be written in terms of the expenditure function as

$$\ln\left[\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right) - 1\right] + \ln\left(\frac{e(\boldsymbol{p},u)}{\pi_2(\boldsymbol{p})}\right) = u\,, \tag{A.92}$$

with $\ln v(\boldsymbol{p},m) = u$. This is a full rank two system that does not satisfy proposition 2.

For the EXP indirect utility function, 0° homogeneity requires that $\mathrm{e}^{B(\boldsymbol{p})} = \pi_1(\boldsymbol{p})$ and $A(\boldsymbol{p})\mathrm{e}^{-B(\boldsymbol{p})} = \pi_2(\boldsymbol{p})^{-(1+\kappa)}$, where again $\pi_1(\boldsymbol{p})$ and $\pi_2(\boldsymbol{p})$ are 1° homogeneous. This model can then be written in terms of the expenditure function as

$$\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right) + \left(\frac{e(\boldsymbol{p},u)}{\pi_2(\boldsymbol{p})}\right)^{\kappa+1} = u\,, \tag{A.93}$$

with $v(\boldsymbol{p},m) = u$. If $\kappa > 0$ and π_1 and π_2 satisfy the conditions given in proposition 2, then this full rank two system is globally regular.

For the TAN indirect utility function, Euler's equation for 0° homogeneity implies that two functions, $\pi_1 : \mathcal{P} \to \mathbb{R}_+$, $\pi_2 : \mathcal{P} \to \mathbb{R}$, exist such that $\pi_1(\boldsymbol{p})$ is 1° homogeneous, $\pi_2(\boldsymbol{p})$ is 0° homogeneous, and

$$v(\boldsymbol{p},m) = \pi_2(\boldsymbol{p})\left[\sin\left(\tau \ln\left(m/\pi_1(\boldsymbol{p})\right)\right) + \cos\left(\tau \ln\left(m/\pi_1(\boldsymbol{p})\right)\right)\right]. \tag{A.94}$$

Set $\alpha(\boldsymbol{p}) = \pi_2(\boldsymbol{p})^{-1}$ to write the TAN model in terms of the expenditure function as

$$\sin\left(\tau \ln\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)\right) + \cos\left(\tau \ln\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)\right) = \alpha(\boldsymbol{p})u\,. \tag{A.95}$$

This is a full rank two system that does not satisfy proposition 2.

Reciprocal Generalized Leontief

The generalized Leontief reciprocal indirect utility function (Barnett and Lee 1985; and Barnett, Lee, and Wolfe 1985) is

$$\frac{1}{v(\boldsymbol{p},m)} = a_0 + \sum_{i=1}^{n} a_i \sqrt{p_i/m} + \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij} \sqrt{p_i p_j}\Big/m \,. \tag{A.96}$$

Define $\pi_1(\boldsymbol{p}) = \left[\sum_{i=1}^{n} a_i \sqrt{p_i}\right]^2$ and $\pi_2(\boldsymbol{p}) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\sqrt{p_i p_j}$, so that we have

$$-\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)^{-½} - \left(\frac{e(\boldsymbol{p},u)}{\pi_2(\boldsymbol{p})}\right)^{-1} = a_0 - \frac{1}{u}\,. \tag{A.97}$$

Define $u = 0$ when $m = 0$ to apply the normalization $\tilde{u} = a_0 - (1/u)$, $d\tilde{u}/du = 1\big/u^2 > 0.$ This is a full rank two system. But neither π_1 nor π_2 as currently defined is globally concave, so that this system does not satisfy proposition 2. However, alternative choices for π_1 and π_2 that are positive valued and globally increasing and concave would give a full rank two, globally regular demand model.

Reciprocal Minflex Laurent

Dropping the exponents on all coefficients in equation (3.1) of Barnett and Lee (1985), the minflex Laurent reciprocal indirect utility function is

$$\begin{aligned}\frac{1}{v(\boldsymbol{p},m)} &= a_0 + \sum_{i=1}^{n} a_i \sqrt{p_i/m} + \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} \sqrt{p_i p_j}\Big/m \\ &+ \sum_{i=1}^{m} c_i\, p_i/m - \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}\, m\Big/\sqrt{p_i p_j}\,.\end{aligned} \tag{A.98}$$

The same definitions for $\pi_1(\boldsymbol{p})$ and $\pi_2(\boldsymbol{p})$, plus the new definitions $\pi_3(\boldsymbol{p}) = \sum_{i=1}^{n} c_i p_i$ and $\pi_4(\boldsymbol{p})^{-1} = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}\Big/\sqrt{p_i p_j}$, imply that

$$-\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)^{-½} - \left(\frac{e(\boldsymbol{p},u)}{\pi_2(\boldsymbol{p})}\right)^{-1} - \left(\frac{e(\boldsymbol{p},u)}{\pi_3(\boldsymbol{p})}\right)^{-1} + \left(\frac{e(\boldsymbol{p},u)}{\pi_4(\boldsymbol{p})}\right) = a_0 - \frac{1}{u}\,. \tag{A.99}$$

Again define $u = 0$ if $m = 0$ and apply the normalization $\tilde{u} = a_0 - (1/u)$. The linear dependence of the two income functions in the middle of the left-hand side implies that this system has rank three and is not full rank four. This is verified by inspection of the budget shares in Barnett, Lee and Wolfe (1985, p. 8, equation (3.4)) and Barnett and Lee (1985, p. 1423). Only π_3 is globally concave, so that proposition 2 is not satisfied.

Reciprocal Indirect Normalized Quadratic

The reciprocal normalized quadratic indirect utility function is (Diewert and Wales 1988)

$$\frac{1}{v(\boldsymbol{p},m)} = b_0 + \left(\frac{\boldsymbol{b}^\top \boldsymbol{p}}{m}\right) + ½\left(\frac{\boldsymbol{p}^\top \boldsymbol{Bp}/m^2}{\boldsymbol{\alpha}^\top \boldsymbol{p}/m}\right) + \boldsymbol{a}^\top \ln\left(\frac{\boldsymbol{p}}{m}\right). \tag{A.100}$$

Since $\pi_1(\boldsymbol{p}) = \boldsymbol{b}^\top \boldsymbol{p}$, $\pi_2(\boldsymbol{p}) = ½\, \boldsymbol{p}^\top \boldsymbol{Bp}/\boldsymbol{\alpha}^\top \boldsymbol{p}$, and $\pi_3(\boldsymbol{p}) \equiv \prod_{i=1}^n p_i^{(a_i/\boldsymbol{a}^\top \boldsymbol{i})}$ are all 1° homogeneous, we can rewrite this in terms of the expenditure function as

$$-\left(\frac{e(\boldsymbol{p},u)}{\pi_1(\boldsymbol{p})}\right)^{-1} - \left(\frac{e(\boldsymbol{p},u)}{\pi_2(\boldsymbol{p})}\right)^{-1} + \boldsymbol{a}^\top \boldsymbol{i} \ln\left(\frac{e(\boldsymbol{p},u)}{\pi_3(\boldsymbol{p})}\right) = b_0 - \frac{1}{u}. \tag{A.101}$$

Define $u = 0$ if $m = 0$ and apply the normalization $\tilde{u} = b_0 - (1/u)$. As can be verified by simple inspection of equation (7) in Diewert and Wales (1988), this system has rank two, and is not full rank three, due to the linear dependence of the first and second income functions on the left-hand side. Moreover, since $\varphi_3(x) = \ln x$ is concave and π_2 is only locally concave, this system does not satisfy proposition 2.

The Rational Rank Four Demand System

The reciprocal indirect utility function for the rational rank four demand system (Lewbel 2003, 2004) is

$$\frac{1}{v(\boldsymbol{p},m)} = \left(\frac{\ln\left[m - a(\boldsymbol{p})\right] - b(\boldsymbol{p})}{c(\boldsymbol{p})}\right)^{-1} + d(\boldsymbol{p}). \tag{A.102}$$

Homogeneity requires that $a(\boldsymbol{p})$ is 1° homogeneous, $b(\boldsymbol{p}) = \ln \pi_1(\boldsymbol{p})$, where $\pi_1(\boldsymbol{p})$ is 1° homogeneous, and $c(\boldsymbol{p})$, $d(\boldsymbol{p})$ are 0° homogeneous. Therefore, rewrite (A.102) as

$$\ln\left(\frac{e(\boldsymbol{p},u)-a(\boldsymbol{p})}{\pi_1(\boldsymbol{p})}\right)=\frac{\sqrt{c(\boldsymbol{p})}u}{\left(-d(\boldsymbol{p})\big/\sqrt{c(\boldsymbol{p})}\right)u+\left(1\big/\sqrt{c(\boldsymbol{p})}\right)}, \qquad \text{(A.103)}$$

so that $\alpha(\boldsymbol{p})=\sqrt{c(\boldsymbol{p})}$, $\beta(\boldsymbol{p})=0$, $\gamma(\boldsymbol{p})=-d(\boldsymbol{p})\big/\sqrt{c(\boldsymbol{p})}$, and $\delta(\boldsymbol{p})=1\big/\sqrt{c(\boldsymbol{p})}$ gives the projective group transformation on the right-hand side, while $f(x)=\ln x$, $\varphi_1(x)=x$, and $\varphi_2(\boldsymbol{p})=-a(\boldsymbol{p})/\pi_1(\boldsymbol{p})$ is independent of income and 0° homogeneous in $\boldsymbol{p}$ on the left-hand side. This system is rank four since $\{1,x\}$ are linearly independent and $f(x)=\ln x$ has nonzero curvature, which combine to produce two linearly independent terms from the left-hand side, while the right-hand side has two independent price indices that define the projective group representation. Because $f(x)=\ln x$ is concave and the right-hand side has full rank two, this system is not globally regular.