CHAPTER 4

The Generalized Quadratic Expenditure System

Jeffrey T. LaFrance* and Rulon D. Pope

Abstract

This chapter presents the indirect preferences for all full rank Gorman and Lewbel demand systems. Each member in this class of demand models is a generalized quadratic expenditure system (GQES). This representation allows applied researchers to choose a small number of price indices and a function of income to specify any exactly aggregable demand system, without the need to revisit the questions of integrability of the demand equations or the implied form and structure of indirect preferences. This characterization also allows for the calculation of exact welfare measures for consumers, either in the aggregate or for specific classes of individuals, and other valuations of interest to applied researchers.

Keywords: aggregation, demand systems, functional form, integrability, rank

JEL classifications: D12, E21

1. Introduction

Specifying the functional form of a system of demand equations is a central focus of empirical economic modeling. Two approaches to this issue are to solve the integrability conditions for a chosen set of demand equations to derive the indirect preference function or to specify the indirect preference

* Corresponding author.

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function directly and then use Roy's Identity or Hotelling's Lemma to generate the demand equations.

The first approach specifies an attractive set of demand equations $\boldsymbol{q}(\boldsymbol{p},m)$ where $\boldsymbol{q}$ is an n-vector of consumption goods, $\boldsymbol{p}$ the associated price vector, and m the income.[1] The most common class of demand models of this type has been the multiplicatively separable and additive form,[2]

$$q_i = \sum_{k=1}^{K} \alpha_{ik}(\boldsymbol{p}) h_k(m), \quad i = 1, \ldots, n, \tag{1}$$

where $\alpha_{ik} : \mathbb{R}^n_{++} \to \mathbb{R}$, $h_k : \mathbb{R}_{++} \to \mathbb{R}$, $\forall\ i = 1, \ldots, n$, $\forall\ k = 1, \ldots, K$. An important strength of this model specification is that it aggregates from micro- to macro-level data. Given a distribution function of income, $F : \mathbb{R}_+ \to [0, 1]$, then we only need K cross-sectional moments,

$$\bar{h}_k = \int h_k(m) dF(m), \quad k = 1, \ldots, K,$$

to estimate Eq. (1) with aggregate data.[3]

Define the $n \times K$ matrix of price functions $\boldsymbol{A}(\boldsymbol{p}) = [\alpha_{ik}(\boldsymbol{p})]$. The *rank* of Eq. (1) is the column rank of $\boldsymbol{A}(\boldsymbol{p})$, with $n \geqslant K$ (Gorman, 1981). *Full rank*

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[1] *Income* is really a nickname for total consumption expenditure.

[2] An important literature on this topic includes: Gorman (1953, 1961, 1965, 1981); Pollak (1969, 1971a, 1971b, 1972); Burt and Brewer (1971); Phlips (1971); Muellbauer (1975, 1976); Cicchetti *et al.* (1976); Howe *et al.* (1979); Deaton and Muellbauer (1980); Jorgenson *et al.* (1980, 1981, 1982); Lau (1982); Russell (1983, 1996); Jorgenson and Slesnick (1984, 1987); Lewbel (1987a, 1988; 1989a, 1989b, 1990, 1991, 2003, 2004); Diewert and Wales (1987, 1988); Blundell (1988); Wales and Woodland (1983); Brown and Walker (1989); van Daal and Merkies (1989); Jorgenson (1990); Pollak and Wales (1969, 1980, 1992); Jerison (1993); Russell and Farris (1993, 1998); and Banks *et al.* (1997). Consistent with this literature, we focus on smooth demand systems with interior solutions.

[3] This property extends to Lau's (1982) *Fundamental Theorem of Exact Aggregation*, where a vector $\boldsymbol{s} \in \mathbb{R}^r$ of demographic or other demand shifters is included in the income functions, the joint distribution function for $(m, \boldsymbol{s})$ is $F(m, \boldsymbol{s})$, and the K cross-sectional moments required for exact aggregation are

$$\bar{h}_k = \int h_k(m, \boldsymbol{s}) dF(m, \boldsymbol{s}), \quad k = 1, \ldots, K.$$

The analysis of homogeneity given below can easily be shown to lead to the Gorman class of functional forms with respect to income. Moreover, 0° homogeneity implies that the $\{h_k\}$ must be multiplicatively separable between income and demographics, that is, $h_k(m, \boldsymbol{s}) = g_k(m) \cdot \ell_k(\boldsymbol{s})\ \forall\ k = 1, \ldots, K$. In other words, 0° homogeneity and Lau's result require that the demand equations are of the form,

$$q_i = \sum_{k=1}^{K} \alpha_{ik}(\boldsymbol{p}) g_k(m) \ell_k(\boldsymbol{s}), \quad i = 1, \ldots, n,$$

with each of the $\{g_k(\mathrm{m})\}$ a member of the Gorman class of functional forms. Hence, all of the results on rank and functional form in Gorman's theory of aggregation hold in this model specification as well.

systems are important because they are *parsimonious*. In parsimonious systems, for any given degree of flexibility in prices and income, the minimum number of parameters needs to be estimated. As a result, the main focus in the literature has been on full rank systems. That is, $\boldsymbol{A}$ has rank K.

Assume that the expenditure function, $e : \mathbb{R}^n_{++} \times \mathbb{R} \to \mathbb{R}_{++}$, defined by

$$e(\boldsymbol{p}, u) \equiv \min_{\boldsymbol{q} \in \mathbb{R}^n_+} \{\boldsymbol{p}^\top \boldsymbol{q} : u(\boldsymbol{q}) \geq u\},$$

and associated with the demand system (Eq. (1)) exists, is smooth, $e \in \mathcal{C}^\infty$, increasing, 1° homogeneous, and concave in $\boldsymbol{p}$, and increasing in u. One difficulty with starting with Eq. (1) is the problem of integrability to well-behaved preferences (Hurwicz and Uzawa, 1971). For this class of models (hereafter a *Gorman system*), the demand system must satisfy 0° homogeneity, adding up, and symmetry and negative semidefiniteness of the Slutsky equations,

$$\begin{aligned} \frac{\partial^2 e}{\partial p_i \partial p_j} &= \sum_{k=1}^K \frac{\partial \alpha_{ik}}{\partial p_j} h_k + \sum_{k=1}^K \alpha_{ik} h'_k \sum_{\ell=1}^K \alpha_{j\ell} h_\ell \\ &= \sum_{k=1}^K \frac{\partial \alpha_{jk}}{\partial p_i} h_k + \sum_{k=1}^K \alpha_{jk} h'_k \sum_{\ell=1}^K \alpha_{i\ell} h_\ell \\ &= \frac{\partial^2 e}{\partial p_j \partial p_i}, \ \forall \ i \neq j. \end{aligned} \tag{2}$$

Each of these properties leads to restrictions on the number of terms, K, the admissible functional forms for the income terms, $\{h_k\}$, the relationships among the terms in the demand model, and the values of the model's parameters. Except for curvature, the role of each of these properties is discussed in detail later.[4] First, two properties are developed that are essential to the identification of the model's parameters during econometric estimation.

2. *A unique representation*

In this section, we discuss the concept of *linear independence* of the price and income functions used throughout this chapter. Let the $n \times K$ matrix of price functions be denoted by $\boldsymbol{A}(\boldsymbol{p}) = [\boldsymbol{\alpha}_1(\boldsymbol{p}) \ \cdots \ \boldsymbol{\alpha}_K(\boldsymbol{p})]$ and let the $K \times 1$ vector of income functions be denoted by $\boldsymbol{h}(m)$. For the system of demand

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[4] However, LaFrance and Pope (2008) discussed and analyzed the local and global monotonicity and concavity properties of the expenditure function for this class of demand models. LaFrance *et al*. (2005, 2006) developed a method to nest aggregable demand models with empirical results that can be economically regular on an open set that contains the convex hull of the data. LaFrance (2008) successfully applied this methodology to U.S. food demand.

equations to have a unique representation on $\mathbb{R}^n_{++} \times \mathbb{R}_{++}$, we need two conditions (Gorman, 1981, pp. 358–359; Russell and Farris, 1998, pp. 201–202).

The first condition needed is that the $\{h_k(m)\}_{k=1}^K$ are linearly independent with respect to the constants in K-dimensional space. That is, there can exist no $\boldsymbol{c} \in \mathbb{R}^K$ satisfying $\boldsymbol{c} \neq \boldsymbol{0}$ and $\boldsymbol{c}^\top \boldsymbol{h}(m^1) = 0\ \forall\ m^1 \in \mathcal{N}(m) \subset \mathbb{R}_{++}$, where $\mathcal{N}(m)$ is an open neighborhood of an arbitrary point in the interior of $\mathcal{M} \subset \mathbb{R}_{++}$, the domain of definition for the $\boldsymbol{h}(m)$. If this is not satisfied, then for any K-vector, $\boldsymbol{d} \in \mathbb{R}^K$, adding the n-vector $\boldsymbol{A}(\boldsymbol{p})\boldsymbol{d}\boldsymbol{c}^\top \boldsymbol{h}(m) \equiv \boldsymbol{0}$ to the system of demands does not change it,

$$\boldsymbol{q} = \boldsymbol{A}(\boldsymbol{p})(\boldsymbol{I} + \boldsymbol{d}\boldsymbol{c}^\top)\boldsymbol{h}(m).$$

We could therefore choose different $\boldsymbol{d}$ vectors to make the matrix $\tilde{\boldsymbol{A}}(\boldsymbol{p}) \equiv \boldsymbol{A}(\boldsymbol{p})(\boldsymbol{I} + \boldsymbol{d}\boldsymbol{c}^\top)$ anything, whereas each such choice is multiplicatively separable between prices and income. That is, the demand system is unidentified and meaningless.

The second condition needed is that the column vectors of $\boldsymbol{A}(\boldsymbol{p})$ are linearly independent with respect to the K-dimensional constants. For this to hold, there can be no $\boldsymbol{c} \in \mathbb{R}^K$ that satisfies $\boldsymbol{c} \neq \boldsymbol{0}$ and $\boldsymbol{A}(\boldsymbol{p}^1)\boldsymbol{c} = \boldsymbol{0}\ \forall\ \boldsymbol{p}^1 \in \mathcal{N}(\boldsymbol{p})$, where in this case $\mathcal{N}(\boldsymbol{p})$ is an open neighborhood of any point in the interior of $\mathcal{P} \subset \mathbb{R}^n_{++}$, the domain of definition for the $n \times K$ array of functions $\boldsymbol{A}(\boldsymbol{p})$. If this property did not hold, then $\forall\ \boldsymbol{d} \in \mathbb{R}^K$, adding $\boldsymbol{A}(\boldsymbol{p})\boldsymbol{c}\boldsymbol{d}^\top \boldsymbol{h}(m) \equiv \boldsymbol{0}$ to the system does not change it,

$$\boldsymbol{q} = \boldsymbol{A}(\boldsymbol{p})(\boldsymbol{I} + \boldsymbol{c}\boldsymbol{d}^\top)\boldsymbol{h}(m).$$

We again could choose any K-vector $\boldsymbol{d}$ to make the n-vector $\tilde{\boldsymbol{h}}(m) \equiv (\boldsymbol{I} + \boldsymbol{c}\boldsymbol{d}^\top)\boldsymbol{h}(m)$ anything while maintaining the multiplicatively separable structure. The demand system would thus again be unidentified and make little sense. We therefore assume throughout that the dimensions of $\boldsymbol{A}$ and $\boldsymbol{h}$ are such that a unique representation exists in all cases.

3. The role of symmetry

The symmetry conditions (Eq. (2)) are identical to those discovered by Sophus Lie (An English translation of Lie's 1880 monograph, with commentary is contained in Hermann, 1975) in his seminal study of transformation groups. Subtracting $\partial^2 e/\partial p_j \partial p_i$ from $\partial^2 e/\partial p_i \partial p_j$, the Slutsky symmetry conditions can be rewritten in terms of $\frac{1}{2}n(n–1)$ vanishing differences,

$$0 = \sum_{k=1}^{K} \left(\frac{\partial \alpha_{ik}}{\partial p_j} - \frac{\partial \alpha_{jk}}{\partial p_i} \right) h_k + \sum_{k=1}^{K} \sum_{\ell=1}^{K} \alpha_{ik}\alpha_{j\ell}(h'_k h_\ell - h_k h'_\ell),\ \forall\, j < i = 2, \ldots, n. \tag{3}$$

In the double sum on the right-hand side, when $k = \ell$, the term $\alpha_{ik}\ \alpha_{jk}$ is multiplied by $h'_k h_k - h_k h'_k = 0$. On the other hand, when $k \neq \ell$, then the term $h'_k h_\ell - h_k h'_\ell$ appears twice, once multiplied by $\alpha_{ik}\alpha_{j\ell}$ and once multiplied by $-\alpha_{i\ell}\alpha_{jk}$. Therefore, we can rewrite Eq. (3) again as a linear system of $\frac{1}{2}n(n–1)$ equations in the $\frac{1}{2}K(K–1)$ terms, $h'_k h_\ell - h_k h'_\ell,\ \ k > \ell$,

$$\begin{aligned} 0 = &\sum_{k=1}^{K}\left(\frac{\partial\alpha_{ik}}{\partial p_j} - \frac{\partial\alpha_{jk}}{\partial p_i}\right)h_k \\ &+ \sum_{k=2}^{K}\sum_{\ell=1}^{k-1}(\alpha_{ik}\alpha_{j\ell} - \alpha_{jk}\alpha_{i\ell})(h'_k h_\ell - h_k h'_\ell),\ j < i = 2,\ldots,n. \end{aligned} \tag{4}$$

Now define the matrices

$$\boldsymbol{B} = \begin{bmatrix} \alpha_{22}\alpha_{11} - \alpha_{12}\alpha_{21} & \cdots & \alpha_{2k}\alpha_{1\ell} - \alpha_{1k}\alpha_{2\ell} & \cdots & \alpha_{2K}\alpha_{1,K-1} - \alpha_{1K}\alpha_{2,K-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{i2}\alpha_{j1} - \alpha_{j2}\alpha_{i1} & \cdots & \alpha_{ik}\alpha_{j\ell} - \alpha_{jk}\alpha_{i\ell} & \cdots & \alpha_{iK}\alpha_{j,K-1} - \alpha_{jK}\alpha_{i,K-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n2}\alpha_{n-1,1} - \alpha_{n-1,2}\alpha_{n1} & \cdots & \alpha_{nk}\alpha_{n-1,\ell} - \alpha_{n-1,k}\alpha_{n,\ell} & \cdots & \alpha_{nK}\alpha_{n-1,K-1} - \alpha_{n-1,K}\alpha_{n,K-1} \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} \dfrac{\partial\alpha_{11}}{\partial p_2} - \dfrac{\partial\alpha_{21}}{\partial p_1} & \cdots & \dfrac{\partial\alpha_{1K}}{\partial p_2} - \dfrac{\partial\alpha_{2K}}{\partial p_1} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial\alpha_{i1}}{\partial p_j} - \dfrac{\partial\alpha_{j1}}{\partial p_i} & \cdots & \dfrac{\partial\alpha_{iK}}{\partial p_j} - \dfrac{\partial\alpha_{jK}}{\partial p_i} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial\alpha_{n,1}}{\partial p_{n-1}} - \dfrac{\partial\alpha_{n-1,1}}{\partial p_n} & \cdots & \dfrac{\partial\alpha_{n,K}}{\partial p_{n-1}} - \dfrac{\partial\alpha_{n-1,K}}{\partial p_n} \end{bmatrix},$$

and the vector[5]

$$\tilde{\boldsymbol{h}} = \left[\, h'_2 h_1 - h_2 h'_1 \quad \ldots \quad h'_k h_\ell - h_k h'_\ell \quad \ldots \quad h'_K h_{K-1} - h_K h'_{K-1} \,\right]^\top.$$

Note that $\boldsymbol{B}$ is $\frac{1}{2}n(n-1) \times \frac{1}{2}K(K-1)$, $\boldsymbol{C}$ is $\frac{1}{2}n(n-1) \times K$, and $\tilde{\boldsymbol{h}}$ is $\frac{1}{2}K(K-1) \times 1$. These definitions allow us to rewrite the symmetry conditions (Eq. (4)) compactly in matrix notation as $\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{C}\boldsymbol{h}$.

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[5] In differential geometry, the terms $h_k(m)h'_\ell(m) - h'_k(m)h_\ell(m),\ \ k \neq \ell$, are called *Jacoby brackets*. When the differential operator, $\partial/\partial m$ is appended to the right of a Jacoby bracket, the result is the *Lie bracket*, $[h_k(m)h'_\ell(m) - h'_k(m)h_\ell(m)]\partial/\partial m$. The K differential operators, $h_k(m)\partial/\partial m,\ \ k = 1,\ldots,K$, forms a finite dimensional system of *vector fields* on the real line and the *Lie algebra* for these vector fields is the linear vector space spanned by the vector fields. The largest Lie algebra on the real line has rank three. The basis $\{\partial/\partial m, m\partial/\partial m, m^2\partial/\partial m\}$ spans this vector space. Russell and Farris (1993) is a very useful introduction to these concepts and their application to Gorman systems. Guillemin and Pollack (1974), Hydon (2000), Olver (1993), and Spivak (1999) are helpful references on differential geometry and applications of Lie's theory to differential equation systems.

If we premultiply both sides of this system of matrix equations by $\boldsymbol{B}^\top$, then we obtain $\boldsymbol{B}^\top \boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{B}^\top \boldsymbol{C}\boldsymbol{h}$. The $\frac{1}{2}K(K-1) \times \frac{1}{2}K(K-1)$ matrix $\boldsymbol{B}^\top \boldsymbol{B}$ is symmetric and positive semidefinite. Therefore, if $\boldsymbol{B}$ has full column rank, then the fundamental rank result of Lie (Hermann, 1975) is $\frac{1}{2}K(K-1) \leq K$. This is equivalent to the condition $K \leq 3$ (Hermann, 1975, pp. 143–146). It also can be shown that the rank of $\boldsymbol{B}$ equals the rank of $\boldsymbol{A}$ (Hermann, 1975, p. 141). Therefore, since $\boldsymbol{B}^\top \boldsymbol{B}$ is of order $\frac{1}{2}K(K-1) \times \frac{1}{2}K(K-1)$ and has the same rank as $\boldsymbol{B}$, which also equals the rank of $\boldsymbol{A}$, it follows that $K \leq 3$ in a full rank Gorman system. This establishes that in a full rank system, *symmetry* leads to demands of the form,

$$q_i = \sum_{k=1}^{3} \alpha_{ij}(\boldsymbol{p}) h_k(m), \quad i = 1, \ldots, n. \tag{5}$$

This important insight was originally stated by Russell (1983) and is explained in detail by Russell and Farris (1993, 1998) and Russell (1996).

A further implication of slutsky symmetry (and symmetry alone; for a detailed discussion see Section 7) for a full rank demand system with the multiplicatively separable and additive structure of Goman is that $\{h_1,\ h_2,\ h_3\}$ in Eq. (5) are related to each other in a fundamental way. From the theory of Lie transformation groups, any full rank demand system with this structure reduces to a special case of a system of Ricatti partial differential equations (Russell, 1983, 1996; Russell and Farris, 1993, 1998),

$$\frac{\partial y}{\partial \boldsymbol{p}} = \tilde{\boldsymbol{\alpha}}_1(\boldsymbol{p}) + \tilde{\boldsymbol{\alpha}}_2(\boldsymbol{p}) y + \tilde{\boldsymbol{\alpha}}_3(\boldsymbol{p}) y^2, \tag{6}$$

where $y = f(e(\boldsymbol{p},\ u))$ is a smooth and strictly monotonic function of expenditure and the $n \times 1$ vectors $\{\tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p})\}$ are derived from the $n \times 1$ vectors $\{\boldsymbol{\alpha}_k(\boldsymbol{p})\}$ in Eq. (1).[6] This expression is derived explicitly during the proof of proposition 3 later.

At this point, however, is it worthwhile to show that all full rank three extended PIGL systems with $f(m) = m^\kappa$, PIGLOG systems with $f(m) = \ln m$, and quadratic expenditure systems (QES) with $f(m) = m$, which are studied by Lewbel (1989a, 1990) and van Daal and Merkies (1989), can be

[6] See Jerison (1993) for an example of a reduced rank system that also has the Gorman structure.

reduced to the compact form,

$$\frac{\partial}{\partial \boldsymbol{p}}\left(\frac{f(e(\boldsymbol{p},u))-\beta_1(\boldsymbol{p})}{\beta_2(\boldsymbol{p})}\right)=\left[\theta(\beta_3(\boldsymbol{p}))+\left(\frac{f(e(\boldsymbol{p},u))-\beta_1(\boldsymbol{p})}{\beta_2(\boldsymbol{p})}\right)^2\right]\frac{\partial \beta_3(\boldsymbol{p})}{\partial \boldsymbol{p}}, \tag{7}$$

where $f(m)\in\{\ln m, m^{\kappa}\}, \beta_1,\beta_2,\beta_3:\mathbb{R}^n_{++}\to\mathbb{R}, \theta:\mathbb{R}\to\mathbb{R}$ and $\beta_1,\beta_2,\beta_3, \theta\in\mathcal{C}^{\infty}$.[7] This is algebraically equivalent to Eq. (6) with the following definitions:

$$\tilde{\boldsymbol{\alpha}}_1(\boldsymbol{p})=\frac{\partial \beta_1(\boldsymbol{p})}{\partial \boldsymbol{p}}+\frac{1}{\beta_2(\boldsymbol{p})}\frac{\partial \beta_2(\boldsymbol{p})}{\partial \boldsymbol{p}}+\left[\frac{\beta_1(\boldsymbol{p})^2}{\beta_2(\boldsymbol{p})}+\theta(\beta_3(\boldsymbol{p}))\beta_2(\boldsymbol{p})\right]\frac{\partial \beta_3(\boldsymbol{p})}{\partial \boldsymbol{p}};$$

$$\tilde{\boldsymbol{\alpha}}_2(\boldsymbol{p})=-\frac{2\beta_1(\boldsymbol{p})}{\beta_2(\boldsymbol{p})}\frac{\partial \beta_3(\boldsymbol{p})}{\partial \boldsymbol{p}};\ \text{and}$$

$$\tilde{\boldsymbol{\alpha}}_3(\boldsymbol{p})=\frac{1}{\beta_2(\boldsymbol{p})}\frac{\partial \beta_3(\boldsymbol{p})}{\partial \boldsymbol{p}}.$$

This establishes a fundamental, poorly known and less well-understood, relationship among these models. That is, the linear expenditure system (LES), QES, almost ideal system (AIS; Deaton and Muellbauer, 1980), quadratic almost ideal system (QAIS; Banks *et al.*, 1997), aggregable translog (Jorgenson *et al.*, 1980, 1981, 1982; Jorgenson and Slesnick, 1984, 1987; Lewbel, 1989b; Jorgenson, 1990), and most other common empirical models are *all* special cases of Eq. (7).

Throughout the discussion here, a bold subscript $\boldsymbol{p}$ denotes a vector of partial derivatives with respect to prices, we use a consistent set of notation to replace the various notations employed in the original articles, and we omit arguments of almost all functions to simplify the notational burden.

In van Daal and Merkies (1989), Eq. (2), group terms in β_2^{-1},

$$\boldsymbol{q}=\beta_2^{-1}(m^2\beta_{3\boldsymbol{p}}+m\beta_{2\boldsymbol{p}}-2m\beta_1\beta_{3\boldsymbol{p}}+\beta_1^2\beta_{3\boldsymbol{p}}-\beta_1\beta_{2\boldsymbol{p}})+\beta_{1\boldsymbol{p}}+\theta\beta_2\beta_{3\boldsymbol{p}}. \tag{8}$$

Regroup terms in the parentheses,

$$\boldsymbol{q}=\beta_2^{-1}\left[(m-\beta_1)^2\beta_{3\boldsymbol{p}}+(m-\beta_1)\beta_{2\boldsymbol{p}}\right]+\beta_{1\boldsymbol{p}}+\theta\beta_2\beta_{3\boldsymbol{p}},$$

[7] This exhausts the set of full rank three nominal income systems with the multiplicatively separable and additive structure of Gorman and real-valued *f*(*m*). LaFrance *et al.* (2005) derive Eq. (6) for last the remaining full rank three case, $f(m)=m^{\iota\tau}$. The symmetry arguments of van Daal and Merkies (1989) applied to this case also leads to Eq. (7). A detailed discussion is presented during the proof of Proposition 3 in Section 7 later. Although the QES is a special case of the extended PIGL model with $\kappa=1$, the complete list of implications implied by Slutsky symmetry were first derived by van Daal and Merkies (1989), which fixed an error in the solution for the indirect preferences of the QES in Howe *et al.* (1979).

gather terms in $\beta_{3\boldsymbol{p}}$, divide by β_2, and isolate the terms involving $\beta_{3\boldsymbol{p}}$ on the right,

$$\frac{\boldsymbol{q}-\beta_{1\boldsymbol{p}}}{\beta_2}-\frac{(m-\beta_1)\beta_{2\boldsymbol{p}}}{\beta_2^2}=\left[\left(\frac{m-\beta_1}{\beta_2}\right)^2+\theta\right]\beta_{3\boldsymbol{p}}. \tag{9}$$

To obtain Eq. (7), note that the left-hand side of Eq. (9) can be written in terms of the expenditure function as

$$\frac{\partial}{\partial \boldsymbol{p}}\left(\frac{e(\boldsymbol{p},u)-\beta_1(\boldsymbol{p})}{\beta_2(\boldsymbol{p})}\right)=\frac{(\boldsymbol{q}-\beta_{1\boldsymbol{p}}(\boldsymbol{p}))}{\beta_2(\boldsymbol{p})}-\frac{(e(\boldsymbol{p},u)-\beta_1(\boldsymbol{p}))\beta_{2\boldsymbol{p}}(\boldsymbol{p})}{\beta_2(\boldsymbol{p})^2}.$$

In Lewbel (1990), case iv, move $\tau m^{\tau-1}$ to the left-hand side, define $\tilde{\beta}_2(\boldsymbol{p})\equiv\beta_2(\boldsymbol{p})^{1/\tau}$ and $\tilde{\beta}_1(\boldsymbol{p})\equiv\beta_1(\boldsymbol{p})/\beta_2(\boldsymbol{p})$,

$$\tau m^{\tau-1}\boldsymbol{q}=\tilde{\beta}_2^\tau\tilde{\beta}_{1\boldsymbol{p}}+\tilde{\beta}_1^2\tilde{\beta}_2^\tau\beta_{3\boldsymbol{p}}+\theta\tilde{\beta}_2^\tau\beta_{3\boldsymbol{p}}+\left(\frac{\tau\tilde{\beta}_{2\boldsymbol{p}}}{\tilde{\beta}_2}-2\tilde{\beta}_1\beta_{3\boldsymbol{p}}\right)m^\tau+\frac{\beta_{3\boldsymbol{p}}}{\tilde{\beta}_2^\tau}m^{2\tau}. \tag{10}$$

Group terms in $\beta_{3\boldsymbol{p}}$, divide by $\tilde{\beta}_2^\tau$, and isolate the terms involving $\beta_{3\boldsymbol{p}}$ on the right,

$$\frac{\tau m^{\tau-1}\boldsymbol{q}}{\tilde{\beta}_2^\tau}-\frac{\tau m^\tau\tilde{\beta}_{2\boldsymbol{p}}}{\tilde{\beta}_2^{\tau+1}}-\tilde{\beta}_{1\boldsymbol{p}}=\left[\left(\frac{m^\tau-\tilde{\beta}_1\tilde{\beta}_2^\tau}{\tilde{\beta}_2^\tau}\right)^2+\theta\right]\beta_{3\boldsymbol{p}}. \tag{11}$$

The left-hand side can be written in terms of the expenditure function as

$$\frac{\partial}{\partial \boldsymbol{p}}\left(\frac{e(\boldsymbol{p},u)^\tau}{\tilde{\beta}_2(\boldsymbol{p})^\tau}-\tilde{\beta}_1(\boldsymbol{p})\right)=\frac{\tau e(\boldsymbol{p},u)^{\tau-1}\boldsymbol{q}}{\tilde{\beta}_2(\boldsymbol{p})^\tau}-\frac{\tau e(\boldsymbol{p},u)^\tau\tilde{\beta}_{2\boldsymbol{p}}(\boldsymbol{p})}{\tilde{\beta}_2(\boldsymbol{p})^{\tau+1}}-\tilde{\beta}_{1\boldsymbol{p}}(\boldsymbol{p}). \tag{12}$$

Redefine $\beta_1(\boldsymbol{p})$ and $\beta_2(\boldsymbol{p})$ as $\beta_1(\boldsymbol{p})\equiv\tilde{\beta}_1(\boldsymbol{p})\tilde{\beta}_2(\boldsymbol{p})^\tau$ and $\beta_2(\boldsymbol{p})\equiv\tilde{\beta}_2(\boldsymbol{p})^\tau$, and substitute these definitions into Eqs. (11) and (12) to obtain Eq. (7).

In Lewbel (1990), case v, fix a sign error and typographical error (see Lewbel (1990, p. 297) to see why these minor corrections are needed) and move $1/m$ to the left,

$$\begin{aligned}\frac{\boldsymbol{q}}{m}&=\frac{\beta_{3\boldsymbol{p}}}{\beta_2}(\ln\beta_1)^2-\frac{\beta_{2\boldsymbol{p}}}{\beta_2}\ln\beta_1+\frac{\beta_{1\boldsymbol{p}}}{\beta_1}+\theta\beta_2\beta_{3\boldsymbol{p}}\\&\quad+\left(\frac{\beta_{2\boldsymbol{p}}-2\beta_{3\boldsymbol{p}}\ln\beta_1}{\beta_2}\right)\ln m+\frac{\beta_{3\boldsymbol{p}}}{\beta_2}(\ln m)^2.\end{aligned} \tag{13}$$

Group terms in $\beta_{3\boldsymbol{p}}$, divide by β_2, and isolate the terms involving $\beta_{3\boldsymbol{p}}$ on the right,

$$\frac{\boldsymbol{q}}{\beta_2 m} - \frac{\beta_{1\boldsymbol{p}}}{\beta_1\beta_2} - \frac{\ln(m/\beta_1)\beta_{2\boldsymbol{p}}}{\beta_2^2} = \left[\left(\frac{\ln(m/\beta_1)}{\beta_2}\right)^2 + \theta\right]\beta_{3\boldsymbol{p}}. \tag{14}$$

To obtain Eq. (7), write the left-hand side of Eq. (14) in terms of the expenditure function as

$$\frac{\partial}{\partial \boldsymbol{p}}\left(\frac{\ln\left[e(\boldsymbol{p},u)/\beta_1(\boldsymbol{p})\right]}{\beta_2(\boldsymbol{p})}\right) = \frac{\boldsymbol{q}}{\beta_2(\boldsymbol{p})e(\boldsymbol{p},u)} - \frac{\beta_{1\boldsymbol{p}}(\boldsymbol{p})}{\beta_1(\boldsymbol{p})\beta_2(\boldsymbol{p})} - \frac{\ln\left[e(\boldsymbol{p},u)/\beta_1(\boldsymbol{p})\right]\beta_{2\boldsymbol{p}}(\boldsymbol{p})}{\beta_2(\boldsymbol{p})^2}.$$

This completes the algebraic derivations that are required to reduce each of these models to the compact form of Eq. (7). Given this unifying representation, a change of variables to $z(\boldsymbol{p},u) = \left[f(e(\boldsymbol{p},u)) - \beta_1(\boldsymbol{p})\right]/\beta_2(\boldsymbol{p})$, simplifies Eq. (7) even further to

$$\frac{\partial z(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \left[\theta(\beta_3(\boldsymbol{p})) + z(\boldsymbol{p},u)^2\right]\frac{\partial \beta_3(\boldsymbol{p})}{\partial \boldsymbol{p}}. \tag{15}$$

This is useful for characterizing the solutions for the indirect preferences of these models. We will return to this result, and make extensive use of it, in Section 7 later.

It is worth emphasizing that Eq. (7) – equivalently, Eq. (15) – follows purely from symmetry. That is, the argument by van Daal and Merkies (1989) leading to their Eq. (2) – equivalently, Eq. (8) – hinges only on symmetry. Also, to obtain his cases iv and v – equivalently, Eqs. (10) and (13) – Lewbel (1990) appeals directly to the results of van Daal and Merkies (1989). In fact, any demand system that reduces to Eq. (6) reduces to Eq. (15). Hence, the solution to this system of Ricatti partial differential equations recovers the indirect preferences for *all* models with the multiplicatively separable and additive structure of Gorman (1981).

4. *The role of homogeneity*

Gorman (1981) noted that the class of models he analyzed is somewhat less interesting because of the restrictions due to the appearance of nominal income in the $\{h_k(m)\}$. In fact, it is shown in the next section that symmetry, 0° homogeneity and adding up, and the fact that the demands are real-valued imply that $y = f(m) \in \{\ln m, m^{\kappa}, m^{\iota\tau}\}$ in Eq. (6), where $\kappa \in \mathbb{R}$, $\tau \in \mathbb{R}_+$, and $\iota = \sqrt{-1}$. This is a severe limitation on the admissible choice of functional form for the income variables in a Gorman system. Lewbel (1989a) notes in a footnote that the essence of the Gorman (1981) restrictions on functional form can be derived purely on the basis of 0° homogeneity. It is worthwhile to demonstrate this fact for a single demand equation.

Proposition 1. *Given the single demand equation with Gorman's multiplicatively separable and additive form,* $q = \sum_{k=1}^{K} \alpha_k(\boldsymbol{p}) h_k(m)$, *for K linearly independent functions of prices and K linearly independent functions of income; then q is* 0° *homogeneous in* $(\boldsymbol{p}, m)$ *only if each income function is either:*

(i) m^{κ}, $\kappa \in \mathbb{R}$;
(ii) $m^{\kappa}(\ln m)^{j}$, $\kappa \in \mathbb{R}$, $j \in \{1, \ldots, K\}$;
(iii) $m^{\kappa}\sin(\tau \ln m)$, $m^{\kappa}\cos(\tau \ln m)$, $\kappa \in \mathbb{R}$, $\tau \in \mathbb{R}_{+}$, *appearing in conjugate pairs with the same* $\{\kappa, \tau\}$ *in each pair; or*
(iv) $m^{\kappa}(\ln m)^{j}\sin(\tau \ln m)$, $m^{\kappa}(\ln m)^{j}\cos(\tau \ln m)$, $\kappa \in \mathbb{R}$, $j \in \{1, \ldots, [\frac{1}{2}K]\}$, $K \geq 4$, *where* $[\frac{1}{2}K]$ *is the largest integer* $\leq \frac{1}{2}K$, *and* $\tau \in \mathbb{R}_{+}$, *appearing in conjugate pairs for each* $\{\kappa, j, \tau\}$ *triple.*

Proof. The Euler equation for 0° homogeneity is

$$\sum_{k=1}^{K} \frac{\partial \alpha_k(\boldsymbol{p})}{\partial \boldsymbol{p}^{\top}} \boldsymbol{p} h_k(m) + \sum_{k=1}^{K} \alpha_k(\boldsymbol{p}) h_k'(m) m = 0. \tag{16}$$

If $K = 1$ and $h_1'(m) = 0$, this reduces to $\partial \alpha_1(\boldsymbol{p})/\partial \boldsymbol{p}^{\top} \boldsymbol{p} = 0$, so that $h_1(m) = c$ and $\alpha_1(\boldsymbol{p})$ is 0° homogeneous. Absorb the constant c into the price index and set $\kappa = 0$ to obtain a special case of (i).

If either $K = 1$ and $h_1'(m) \neq 0$ or $K \geq 2$, then neither sum in Eq. (16) can vanish without contradicting the linear independence of the $\{\alpha_k(\boldsymbol{p})\}$ or the $\{h_k(m)\}$. In this case, write the Euler equation as

$$\frac{\sum_{k=1}^{K} \alpha_k(\boldsymbol{p}) h_k'(m) m}{\sum_{k=1}^{K} \left[\partial \alpha_k(\boldsymbol{p})/\partial \boldsymbol{p}^{\top} \boldsymbol{p}\right] h_k(m)} = -1.$$

Since the right-hand side is constant, we must be able to recombine the left-hand side to be independent of both $\boldsymbol{p}$ and m. In other words, the terms in the numerator must recombine in some way so that it is proportional to the denominator, with –1 as the proportionality factor. Clearly, if these two functions are proportional, their functional forms must be the same. Linear independence of the $\{h_k(m)\}$ then implies that each $h_k'(m)m$ must be a linear function of the $\{h_k(m)\}$ with constant coefficients,

$$h_k'(m)m = \sum_{\ell=1}^{K} c_{k,\ell} h_{\ell}(m), \quad k = 1, \ldots, K. \tag{17}$$

This is a complete system of K linear, homogeneous, ordinary differential equations (odes), of the form commonly known as Cauchy's linear differential equation. To prove the proposition, first we convert Eq. (17) into a system of linear odes with constant coefficients through a change of variables from m to $x = \ln m$ (Cohen, 1933, pp. 124–125). Then we identify the complete set of solutions for this new system of odes.

Since $m(x) = e^x$ and $m'(x) = m(x)$, defining $\tilde{h}_k(x) \equiv h_k(m(x)), k = 1, \ldots, K$, and applying this change of variables yields

$$\tilde{h}'_k(x) = \sum_{\ell=1}^{K} c_{k,\ell} \tilde{h}_\ell(x), \quad k = 1, \ldots, K. \tag{18}$$

In matrix form, this system of linear, first-order, homogeneous odes is $\tilde{\boldsymbol{h}}'(x) - \boldsymbol{C}\tilde{\boldsymbol{h}}(x) = 0$, and the characteristic equation is $|\boldsymbol{C} - \lambda \boldsymbol{I}| = 0$. This is a Kth order polynomial in λ, for which the fundamental theorem of algebra (Gauss, 1799) implies that there are exactly K roots. Some of these roots may repeat and some may be complex conjugate pairs. Let the characteristic roots be denoted by $\lambda_k,\ \ k = 1, \ldots, K$.

By repeated differentiation and substitution of any one of the odes in Eq. (18), the system of K first-order odes is equivalent to a single linear homogeneous ode of order K. The general solution to a linear homogeneous ode of order K is the sum of K linearly independent particular solutions (Cohen, 1933, Chapter 6; Boyce and DiPrima, 1977, Chapter 5).[8]

Let there be $R \geq 0$ roots that repeat and reorder the income functions as necessary in the following way. Label the first repeating root (if one exists) as λ_1 and let its multiplicity be denoted by $M_1 \geq 1$. Let the second repeating root (if one exists) be the $M_1 + 1$st root. Label this root as λ_2 and its multiplicity as $M_2 \geq 1$. Continue in this manner until there are no more repeating roots. Let the total number of repeated roots be $M = \sum_{k=1}^{R} M_k$. Label the remaining $K - M \geq 0$ unique roots as λ_k for each $K = M + 1, \ldots, K$. Then the general solution to Eq. (18) can be written as

$$\tilde{h}_k(x) = \sum_{r=1}^{R} \left[\sum_{\ell=1}^{M_r} d_{k\ell} x^{(\ell-1)} e^{\lambda_r x} \right] + \sum_{\ell=M+1}^{K} d_{k\ell} e^{\lambda_\ell x}, \quad k = 1, \ldots, K. \tag{19}$$

Substitute Eq. (19) into the demand equation for q to obtain

$$\begin{aligned} q &= \sum_{k=1}^{K} \alpha_k(\boldsymbol{p}) \left[\sum_{r=1}^{R} \sum_{\ell=1}^{M_r} d_{k\ell} m^{\lambda_r} (\ln m)^{(\ell-1)} + \sum_{\ell=M+1}^{K} d_{k\ell} m^{\lambda_\ell} \right] \\ &= \sum_{r=1}^{R} \sum_{\ell=1}^{M_r} \left[\sum_{k=1}^{K} d_{k\ell} \alpha_k(\boldsymbol{p}) \right] m^{\lambda_r} (\ln m)^{(\ell-1)} + \sum_{\ell=M+1}^{K} \left[\sum_{k=1}^{K} d_{k\ell} \alpha_k(\boldsymbol{p}) \right] m^{\lambda_\ell} \\ &\equiv \sum_{r=1}^{R} \sum_{k=1}^{M_r} \tilde{\alpha}_{kr}(\boldsymbol{p}) m^{\lambda_r} (\ln m)^{(k-1)} + \sum_{k=M+1}^{K} \tilde{\alpha}_k(\boldsymbol{p}) m^{\lambda_k}. \end{aligned} \tag{20}$$

[8] Here, linear independence of the K functions, $\{f_1, \ldots, f_K\}$ of the variable x means that there is no nonvanishing K-vector, $(\alpha_1, \ldots, \alpha_K)$ such that $a_1 f_1 + \cdots + a_K f_K = 0$ for all values of the variables in an open neighborhood of any point $[x, f_1(x), \ldots, f_K(x)]$. Cohen (1933, pp. 303–306), gives necessary and sufficient conditions for this property.

The terms in the first double sum generate cases (i) and (ii), and by de Moivre's theorem ($e^{\pm \imath x} = \cos(x) \pm \imath \sin(x)\ \forall\ x \in \mathbb{R}$), case (iv) if $K \geq 4$ and a pair of complex conjugate roots repeats. The terms in the sum on the far right give case (i) for unique real roots and, again by de Moivre's theorem, case (iii) for unique pairs of complex conjugate roots. ■

Note that this result on the set of admissible functional forms hinges entirely on 0° homogeneity – and not on symmetry or adding up. Because only one demand equation is analyzed, neither symmetry nor adding up applies to the aforementioned argument. This result is crucial to understanding one of the key properties of Gorman systems. It is because of the multiplicatively separable and additive structure of a Gorman system defined in terms of nominal income that 0° homogeneity in $(\boldsymbol{p}, m)$ only can be achieved by multiplication (through power functions) or addition (through logarithmic functions).

5. *The role of adding up*

Recalling Eq. (1), applying adding up to a Gorman system implies

$$m = \sum_{k=1}^{K} \boldsymbol{p}^{\top} \boldsymbol{\alpha}_k(\boldsymbol{p}) h_k(m) = \sum_{k=1}^{K} a_k h_k(m),$$

where the $\{a_k\}_{k=1}^{K}$ are absolute constants, independent of $(\boldsymbol{p}, m)$, because the function on the left is identically m for all $\boldsymbol{p} \in \mathcal{P}$. Linear independence of the $\{h_k\}$ therefore implies that *one and only one* income function is the identity, $h_k(m) \equiv m$, the associated vector of price functions satisfies $\boldsymbol{p}^{\top} \boldsymbol{\alpha}_k(\boldsymbol{p}) \equiv 1$, and all other vectors of price functions must satisfy $\boldsymbol{p}^{\top} \boldsymbol{\alpha}_{\ell}(\boldsymbol{p}) \equiv 0,\ \ \ell \neq k$. This is a special case of the restrictions on functional form due to 0° homogeneity. The reason for this added restriction is that the expenditure function is 1° homogeneous in $\boldsymbol{p}$. This implies both adding up and, by the derivative property of a 1° homogeneous function, 0° homogeneity of Marshallian demands in $(\boldsymbol{p}, m)$ and Hicksian demands in $\boldsymbol{p}$. Since 0° homogeneity is not sufficient for adding up, one added restriction on the functional forms for the income terms is implied when adding up is imposed on top of 0° homogeneity.

A final restriction on the functional forms for the income terms is a consequence of Slutsky symmetry, 0° homogeneity, adding up, and the fact that demands are real-valued, all taken together. In particular, when income is raised to a power, the exponent either must be purely real, m^{κ}, or purely complex, $m^{\imath\tau}$. To see this, consider a full rank three system that has been reduced by symmetry to

$$\begin{aligned} f'(m)\boldsymbol{q} =& [\boldsymbol{\alpha}_0(\boldsymbol{p}) + \imath \boldsymbol{\alpha}_1(\boldsymbol{p})] + \left[\boldsymbol{\beta}_0(\boldsymbol{p}) + \imath \boldsymbol{\beta}_1(\boldsymbol{p})\right] f(m) \\ &+ \left[\boldsymbol{\gamma}_0(\boldsymbol{p}) + \imath \boldsymbol{\gamma}_1(\boldsymbol{p})\right] f(m)^2, \end{aligned} \tag{21}$$

where $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1 : \mathcal{P} \to \mathbb{R}^n$ and $f(m) = m^{\kappa+\imath\tau}$.[9]

Because $f'(m) = (\kappa + \imath\tau)m^{\kappa+\imath\tau-1}$, substitute $m^{\kappa+\imath\tau}$ for $f(m)$ and $(\kappa + \imath\tau)m^{\kappa+\imath\tau-1}$ for $f'(m)$ in Eq. (21) and solve for the vector of quantities demanded to obtain

$$\boldsymbol{q} = \left(\frac{\boldsymbol{\alpha}_0(\boldsymbol{p}) + \imath\boldsymbol{\alpha}_1(\boldsymbol{p})}{\kappa + \imath\tau}\right) m^{1-(\kappa+\imath\tau)} + \left(\frac{\boldsymbol{\beta}_0(\boldsymbol{p}) + \imath\boldsymbol{\beta}_1(\boldsymbol{p})}{\kappa + \imath\tau}\right) m + \left(\frac{\boldsymbol{\gamma}_0(\boldsymbol{p}) + \imath\boldsymbol{\gamma}_1(\boldsymbol{p})}{\kappa + \imath\tau}\right) m^{1+(\kappa+\imath\tau)}.$$

Applying de Moivre's theorem then yields

$$\begin{aligned}\boldsymbol{q} = {} & \left(\frac{\boldsymbol{\beta}_0(\boldsymbol{p}) + \imath\boldsymbol{\beta}_1(\boldsymbol{p})}{\kappa + \imath\tau}\right) m \\ & + \left[\left(\frac{\boldsymbol{\alpha}_0(\boldsymbol{p}) + \imath\boldsymbol{\alpha}_1(\boldsymbol{p})}{\kappa + \imath\tau}\right) m^{1-\kappa} + \left(\frac{\boldsymbol{\gamma}_0(\boldsymbol{p}) + \imath\boldsymbol{\gamma}_1(\boldsymbol{p})}{\kappa + \imath\tau}\right) m^{1+\kappa}\right] \cos(\tau \ln m) \\ & + \imath\left[\left(\frac{\boldsymbol{\gamma}_0(\boldsymbol{p}) + \imath\boldsymbol{\gamma}_1(\boldsymbol{p})}{\kappa + \imath\tau}\right) m^{1+\kappa} - \left(\frac{\boldsymbol{\alpha}_0(\boldsymbol{p}) + \imath\boldsymbol{\alpha}_1(\boldsymbol{p})}{\kappa + \imath\tau}\right) m^{1-\kappa}\right] \sin(\tau \ln m).\end{aligned} \tag{22}$$

Therefore, for the demands to be real-valued, each vector of price functions in Eq. (22) must have real elements.

First, this implies $\boldsymbol{\beta}_0 + \imath\boldsymbol{\beta}_1 = (\kappa + \imath\tau)\boldsymbol{\beta}$ for some $\boldsymbol{\beta} : \mathcal{P} \to \mathbb{R}^n$ such that $\boldsymbol{p}^\top \boldsymbol{\beta}(\boldsymbol{p}) \equiv 1$. Second, neither $\boldsymbol{\alpha}_0 + \imath\boldsymbol{\alpha}_1 = (\kappa + \imath\tau)\boldsymbol{\alpha}$ nor $\boldsymbol{\gamma}_0 + \imath\boldsymbol{\gamma}_1 = (\kappa + \imath\tau)\boldsymbol{\gamma}$ can be true for *any* pair of vector-valued functions, $\boldsymbol{\alpha}, \boldsymbol{\gamma} : \mathcal{P} \to \mathbb{R}^n$. Otherwise, the elements in the vector of price functions that premultiplies $\sin(\tau \ln m)$ are complex-valued. Therefore, if $\tau \neq 0$, then $\kappa = 0$, since $\sin(\tau \ln m)$ and $\cos(\tau \ln m)$ are linearly independent $\forall\ \tau \neq 0$. Conversely, if $\kappa \neq 0$, then $\tau = 0$, since $m^{1-\kappa}$ and $m^{1+\kappa}$ are linearly independent $\forall\ \kappa \neq 0$. Third, if $\tau \neq 0$ (so that $\kappa = 0$), then since $1/\imath = -\imath$ and $\imath^2 = -1$, it follows that $\boldsymbol{\alpha}_1 + \boldsymbol{\gamma}_1 - \imath(\boldsymbol{\alpha}_0 + \boldsymbol{\gamma}_0)$ and $-\boldsymbol{\alpha}_0 + \boldsymbol{\gamma}_0 - \imath(\boldsymbol{\alpha}_1 - \boldsymbol{\gamma}_1)$ must have real elements, identically in $\boldsymbol{p}$. This implies $\boldsymbol{\gamma}_0 = -\boldsymbol{\alpha}_0$ and $\boldsymbol{\gamma}_1 = \boldsymbol{\alpha}_1$, so that

$$\boldsymbol{q} = \tilde{\boldsymbol{\alpha}}_0(\boldsymbol{p})m + \tilde{\boldsymbol{\alpha}}_1(\boldsymbol{p})m \cos(\tau \ln m) + \tilde{\boldsymbol{\alpha}}_2(\boldsymbol{p})m \sin(\tau \ln m),$$

where $\tilde{\boldsymbol{\alpha}}_0 \equiv \boldsymbol{\beta}, \tilde{\boldsymbol{\alpha}}_1 = 2\boldsymbol{\alpha}_1/\tau$, and $\tilde{\boldsymbol{\alpha}}_2 = -2\boldsymbol{\alpha}_0/\tau$. This is the trigonometric functional form found by Gorman (1981), for which the indirect utility function was obtained by Lewbel (1988, 1990). Since the sine and cosine functions are periodic, with a complete period on the interval $[0, 2\pi]$, no loss in generality results from restricting τ to be nonnegative. However,

[9] It is sufficient to consider full rank three for our purposes since complex roots always appear as conjugate pairs and the maximum rank of any Gorman system is three. However, Gorman (1981) shows that this property holds for all rank three systems, including those with reduced rank and $K > 3$. The argument here can be extended to the reduced rank case through careful attention to several technical details. It also can be shown that the specification for the complex-valued vectors of price functions in Eq. (21) is without loss of generality.

$\kappa = 0$ is possible in the case of a purely real exponent, and nothing mathematically precludes either a positive or negative value of κ.

Summarizing, the following results have been obtained: (1) the reduction of any full rank Gorman system to a system of polynomial partial differential equations that is at most quadratic in $f(m)$ is due to *Slutsky symmetry*; (2) the restriction on the functional form of $f(m)$ to logarithmic and power functions is due to 0° *homogeneity*; (3) the restriction that one income function is m is due to *adding up*; and (4) if $f(m)$ is a power function, then the restriction that the exponent must be purely real or purely complex is jointly due to *symmetry,* 0° *homogeneity, adding up, and real-valued demands.*

6. Deflated income systems

In response to the functional form restrictions found by Gorman (1981), Lewbel (1989a) introduced the deflated income Gorman system (hereafter a *Lewbel system*),

$$\boldsymbol{q} = \sum_{k=1}^{K} \boldsymbol{\alpha}_k(\boldsymbol{p}) h_k(e(\boldsymbol{p}, u)/\pi(\boldsymbol{p})), \tag{23}$$

with $\pi : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$, $\pi \in \mathcal{C}^\infty$, strictly positive-valued, increasing, 1° homogeneous, and weakly concave in $\boldsymbol{p}$. This structure maintains exact aggregation in deflated income. That is, the real moments of income can be used to estimate aggregate demand functions.

To relate Lewbel systems to Gorman systems, first note that adding up implies

$$m \equiv \sum_{k=1}^{K} \boldsymbol{p}^\top \boldsymbol{\alpha}_k(\boldsymbol{p}) h_k\big(m/\pi(\boldsymbol{p})\big).$$

As a result, linear independence of the $\{h_k\}$ implies that one and only one must be $m/\pi(\boldsymbol{p})$ and the associated vector of price functions must be $\partial\pi(\boldsymbol{p})/\partial\boldsymbol{p}$. WLOG, let this be the first one, and bring it to the left-hand side of Eq. (23) to obtain,

$$\boldsymbol{q} - \frac{m}{\pi(\boldsymbol{p})}\frac{\partial\pi(\boldsymbol{p})}{\partial\boldsymbol{p}} = \sum_{k=2}^{K} \boldsymbol{\alpha}_k(\boldsymbol{p}) h_k(m/\pi(\boldsymbol{p})). \tag{24}$$

Defining the deflated expenditure function by $\tilde{e}(\boldsymbol{p}, u) \equiv e(\boldsymbol{p}, u)/\pi(\boldsymbol{p})$, it follows that

$$\frac{\partial\tilde{e}(\boldsymbol{p}, u)}{\partial\boldsymbol{p}} = \sum_{k=2}^{K} \tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p}) h_k(\tilde{e}(\boldsymbol{p}, u)), \tag{25}$$

where $\tilde{\boldsymbol{\alpha}}_k \equiv \boldsymbol{\alpha}_k/\pi,\ \ k = 2, \ldots, K$. This is multiplicatively separable between $\boldsymbol{p}$ and $\tilde{e}$ and has the additive structure of a Gorman system. In this system, however, the only issue is symmetry because both 0° homogeneity and adding up are satisfied as long as the vectors of price functions satisfy $\boldsymbol{p}^\top \tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p}) = 0$ and $\tilde{\boldsymbol{\alpha}}_k(\lambda \boldsymbol{p}) \equiv \tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p})\ \forall \lambda > 0,\ \forall\ k = 2, \ldots, K$.

Hence, applying Lie (Hermann, 1975) to a full rank Lewbel system reduces it to the Ricatti equations in Eq. (6) but now with $y(\boldsymbol{p}, u) \equiv f(\tilde{e}(\boldsymbol{p}, u)), f \in \mathcal{C}^\infty$, and $f'(\tilde{e}) \neq 0$. A Lewbel system can achieve rank four and the restriction on the functional form of $f(\tilde{e})$ now has been eliminated. These properties are discussed in more detail later.

7. The common structure of Gorman and Lewbel systems

Ricatti partial differential equations of the form Eq. (6) have been studied extensively in the mathematical theory of differential equations. Recalling Eq. (15), a key property of all full rank Gorman and Lewbel systems is the following.

Proposition 2. *Let* $z : \mathbb{R}^n_{++} \times \mathbb{R} \to \mathbb{R}$, $\theta : \mathbb{R} \to \mathbb{R}$, *and* $\eta : \mathbb{R}^n_{++} \to \mathbb{R}_+$, $z, \theta, \eta \in \mathcal{C}^\infty$, *satisfy Eq.* (15) *with* $\partial\eta(\boldsymbol{p})/\partial\boldsymbol{p} \neq 0$. *Then* $z(\boldsymbol{p}, u) \equiv w(\eta(\boldsymbol{p}), u)$, *with* $w(x, u)$ *satisfying the partial differential equation* $\partial w(x, u)/\partial x = \theta(x) + w(x, u)^2$.

Proof. Differentiate both sides of the system of partial differential equations,

$$\partial z(\boldsymbol{p}, u)/\partial \boldsymbol{p} = \left[\theta(\eta(\boldsymbol{p})) + z(\boldsymbol{p}, u)^2\right] \partial\eta(\boldsymbol{p})/\partial\boldsymbol{p},$$

with respect to $\boldsymbol{p}^\top$ to obtain,

$$\frac{\partial^2 z(\boldsymbol{p}, u)}{\partial\boldsymbol{p}\partial\boldsymbol{p}^\top} = \theta'(\eta(\boldsymbol{p}))\frac{\partial\eta(\boldsymbol{p})}{\partial\boldsymbol{p}}\frac{\partial\eta(\boldsymbol{p})}{\partial\boldsymbol{p}^\top} + \left[\theta(\eta(\boldsymbol{p})) + z(\boldsymbol{p}, u)^2\right]\frac{\partial^2\eta(\boldsymbol{p})}{\partial\boldsymbol{p}\partial\boldsymbol{p}^\top} + 2z(\boldsymbol{p}, u)\frac{\partial z(\boldsymbol{p}, u)}{\partial\boldsymbol{p}}\frac{\partial\eta(\boldsymbol{p})}{\partial\boldsymbol{p}^\top}.$$

Hence, $(\partial z/\partial\boldsymbol{p}) \times (\partial\eta/\partial\boldsymbol{p})^\top$ is symmetric, which implies $z(\boldsymbol{p}, u) = w(\eta(\boldsymbol{p}), u)$ (Goldman and Uzawa, 1964, Lemma 1).

Now differentiate the separable function with respect to $\boldsymbol{p}$ to obtain,

$$\frac{\partial z(\boldsymbol{p}, u)}{\partial\boldsymbol{p}} = \frac{\partial w(\eta(\boldsymbol{p}), u)}{\partial\eta} \cdot \frac{\partial\eta(\boldsymbol{p})}{\partial\boldsymbol{p}} = \left[\theta(\eta(\boldsymbol{p})) + w(\eta(\boldsymbol{p}), u)^2\right]\frac{\partial\eta(\boldsymbol{p})}{\partial\boldsymbol{p}},$$

which together with $\partial\eta(\boldsymbol{p})/\partial\boldsymbol{p} \neq \boldsymbol{0}$ implies $\partial w(x, u)/\partial x = \theta(x) + w(x, u)^2$. ■

The formal mathematical definition of $w : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is

$$w(\eta(\boldsymbol{p}), u) = \begin{cases} u, & \text{if } K = 1, 2, \text{ or } K = 3 \text{ or } 4 \text{ and } \theta'(x) = 0, \\ u + \int_0^{\eta(\boldsymbol{p})}[\theta(x) + w(x, u)^2]dx, & \text{if } K = 3 \text{ or } 4 \text{ and } \theta'(x) \neq 0, \end{cases} \tag{26}$$

subject to $w(0,u)=u$ and $\partial w(0,u)/\partial x=\theta(0)+u^2$.[10] The function w plays an important role in the indirect preferences for *all* Gorman and Lewbel systems. We can now prove the following result (LaFrance and Pope, 2008a, 2008b).

Proposition 3. *Let* $\pi : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$, $\pi \in \mathcal{C}^\infty$, *be strictly positive-valued,* 1° *homogeneous, increasing, and concave; let* $\eta : \mathbb{R}^n_{++} \to \mathbb{R}_{+}, \eta \in \mathcal{C}^\infty$, *be positive-valued,* 0° *homogeneous; let* $\alpha, \beta, \gamma, \delta : \mathbb{R}^n_{++} \to \mathbb{C} = \{x + \imath y,\ x, y \in \mathbb{R}\}, \alpha, \beta, \gamma, \delta \in \mathcal{C}^\infty$, *be* 0° *homogeneous, satisfying* $\alpha\delta - \beta\gamma \equiv 1$; *and let* $f : \mathbb{R} \to \mathbb{C}, f \in \mathcal{C}^\infty, f' \neq 0$. *Then the expenditure function for any full rank Gorman or Lewbel system exists if and only if it is a special case of*

$$e(\boldsymbol{p}, u) = \pi(\boldsymbol{p}) \cdot f^{-1}\left(\frac{\alpha(\boldsymbol{p})w(\eta(\boldsymbol{p}), u) + \beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})w(\eta(\boldsymbol{p}), u) + \delta(\boldsymbol{p})}\right), \tag{27}$$

where $w(\eta(\boldsymbol{p}), u)$ *is defined by Eq.* (26).

Proof. First consider sufficiency by differentiating Eq. (27) and applying Hotelling's lemma. To make the notation as compact as possible, let a bold subscript $\boldsymbol{p}$ denote a vector of partial derivatives with respect to prices, and suppress prices and the utility index as arguments to yield (after considerable algebra),

$$\begin{aligned}\frac{\partial f(\tilde{e})}{\partial \boldsymbol{p}} &= f'(\tilde{e})\left[\left(\frac{\boldsymbol{q}}{\pi}\right) - \left(\frac{\pi_{\boldsymbol{p}}}{\pi}\right)\tilde{e}\right] = \left[\alpha\beta_{\boldsymbol{p}} - \beta\alpha_{\boldsymbol{p}} + (\alpha^2\theta + \beta^2)\eta_{\boldsymbol{p}}\right] \\ &\quad + \left[\beta\gamma_{\boldsymbol{p}} - \gamma\beta_{\boldsymbol{p}} + \delta\alpha_{\boldsymbol{p}} - \alpha\delta_{\boldsymbol{p}} - 2(\alpha\gamma\theta + \beta\delta)\eta_{\boldsymbol{p}}\right] f(\tilde{e}) \\ &\quad + \left[\gamma\delta_{\boldsymbol{p}} - \delta\gamma_{\boldsymbol{p}} + (\alpha^2\theta + \delta^2)\eta_{\boldsymbol{p}}\right] f(\tilde{e})^2.\end{aligned} \tag{28}$$

This has precisely the quadratic structure of Eq. (6) with appropriate definitions for each of the vector-valued price functions. Thus, the representation given by the proposition generates demand systems that have the multiplicatively separable and additive structure of Gorman and Lewbel demand systems.

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[10] The change of variables $w(x,u) = -\partial v(x,u)/\partial x / v(x,u)$ converts the Riccati partial differential equation in w to a linear second-order differential equation $\partial^2 v(x,u)/\partial x^2 + \theta(x)v(x,u) = 0$. This requires two initial conditions. The two chosen here are a convenient normalization for the utility index and guarantee smoothness of w at $x = 0$ for all u. Linear, second-order differential equations with non-constant coefficients generally do not have simple solutions. However, a convergent infinite series of simple functions can be found in many cases (Boyce and DiPrima, 1977, Chapter 4).

The first line of Eq. (26) is a normalization of the utility index that can be made WLOG when $K = 1, 2$ and when $K = 3, 4$ and θ is constant. Note that $\theta \neq 0$, whether constant or not, only can occur in a Gorman system if $K = 3$ and in a Lewbel system if $K = 4$.

Next, make the substitution $m/\pi = \tilde{e}$ and rearrange Eq. (28) to solve for the vector of quantities demanded on the left-hand side to obtain

$$\begin{aligned} \boldsymbol{q} = \pi_{\boldsymbol{p}} \times \left(\frac{m}{\pi}\right) + \pi\Bigg\{ & \left[\alpha\beta_{\boldsymbol{p}} - \beta\alpha_{\boldsymbol{p}} + (\alpha^2\theta + \beta^2)\eta_{\boldsymbol{p}}\right]\frac{1}{f'(m/\pi)} \\ & + \left[\beta\gamma_{\boldsymbol{p}} - \gamma\beta_{\boldsymbol{p}} + \delta\alpha_{\boldsymbol{p}} - \alpha\delta_{\boldsymbol{p}} - 2(\alpha\gamma\theta + \beta\delta)\eta_{\boldsymbol{p}}\right]\frac{f(m/\pi)}{f'(m/\pi)} \\ & + \left[\gamma\delta_{\boldsymbol{p}} - \delta\gamma_{\boldsymbol{p}} + (\alpha^2\theta + \delta^2)\eta_{\boldsymbol{p}}\right]\frac{f(m/\pi)^2}{f'(m/\pi)}\Bigg\}. \end{aligned} \tag{29}$$

Note that there are a total of four income terms on the right-hand side of Eq. (29) with four associated price function vectors and both groups of four functions can be linearly independent. This implies a maximum rank of four. But, defining $\tilde{m} = m/\pi$, whenever $f(\tilde{m}) \in \{\ln \tilde{m}, \tilde{m}^{\kappa}, \tilde{m}^{\iota\tau}\}$, either $f(\tilde{m})/f'(\tilde{m})$ or $1/f'(\tilde{m})$ is proportional to $\tilde{m}$. Because the first term on the right is automatically proportional to $\tilde{m}$, such a choice for f reduces the number of linearly independent income functions by one, resulting in a maximum rank of three in a Gorman system. Thus, a Lewbel system has rank equal to one plus the rank of an otherwise identical Gorman system if and only if $f(\tilde{m}) \notin \{\ln \tilde{m}, \tilde{m}^{\kappa}, \tilde{m}^{\iota\tau}\}$; that is, it is *not* one of the functional forms found by Gorman. This shows precisely how rank can increase by one additional linearly independent vector of price functions and one linearly independent income function in a Lewbel system. Thus, the rank of any demand system obtained from the set of expenditure functions defined in the proposition is at most four for an arbitrary choice for f, and at most three whenever f is chosen to be a member of the Gorman class of functional forms.

To prove necessity, the representation result for all Gorman systems is derived first, followed by all Lewbel systems.

7.1. Gorman systems

Full rank one systems can always be written as $e(\boldsymbol{p}, u)/\pi(\boldsymbol{p}) = u$, $\pi(\boldsymbol{p})$ positive valued, 1° homogeneous, increasing, and concave due to adding up and ordinal utility, which together imply that $f(m) = m$, WLOG.

From the results of Muellbauer (1975, 1976), we know that any full rank two Gorman system must be a PIGL (i.e., $f(m) = m^{\kappa}$) or a PIGLOG (i.e., $f(m) = \ln m$) demand model. For the full rank two PIGL model, we have

$$v(\boldsymbol{p}, m) = \frac{[m^{\kappa} - \beta_1(\boldsymbol{p})]}{\beta_2(\boldsymbol{p})},$$

with $\beta_1(\boldsymbol{p})$ and $\beta_2(\boldsymbol{p})$ κ° homogeneous. Rewrite this in terms of deflated expenditure,

$$\left(\frac{e(\boldsymbol{p}, u)}{\pi(\boldsymbol{p})}\right)^{\kappa} = u + \beta(\boldsymbol{p}),$$

with $\pi(\boldsymbol{p}) \equiv \beta_2(\boldsymbol{p})^{1/\kappa}$ 1° homogeneous, $\beta(\boldsymbol{p}) \equiv \beta_1(\boldsymbol{p})/\beta_2(\boldsymbol{p})$ 0° homogeneous, and the implicit definitions $\alpha = \delta = 1$ and $\gamma = 0$ to obtain Eq. (27).

For the full rank two PIGLOG model, we have

$$v(\boldsymbol{p}, m) = \frac{[\ln m - \beta_1(\boldsymbol{p})]}{\beta_2(\boldsymbol{p})},$$

where $\beta_1(\boldsymbol{p}) = \ln \tilde{\beta}_1(\boldsymbol{p})$, with $\tilde{\beta}_1(\boldsymbol{p})$ 1° homogeneous, $\beta_2(\boldsymbol{p})$ 0° homogeneous. Rewrite this in terms of deflated expenditure,

$$\ln\left(\frac{e(\boldsymbol{p}, u)}{\tilde{\beta}_1(\boldsymbol{p})}\right) = \beta_2(\boldsymbol{p})u.$$

Define $\alpha = \sqrt{\beta_2}$, $\beta = \gamma = 0$, $\delta = 1/\sqrt{\beta_2}$, and $\pi = \tilde{\beta}_1$ to obtain Eq. (27).

For a full rank three system, three functional forms, $f(m) \in \{\ln m, m^{\kappa}, m^{\iota\tau}\}$, and four cases for θ must be considered, $\theta(x) \equiv \lambda$, a positive, zero, or negative constant, and $\theta'(x) \neq 0$. When $f(m) \in \{m^{\kappa}, \ln m\}$, $\kappa \in \mathbb{R}$, and $\theta(x) \equiv \lambda$, a constant, the van Daal and Merkies (1989) and Lewbel (1987, 1990) implicit solution for indirect preferences is

$$\int^{-\beta_3(\boldsymbol{p})/\left[f(e(\boldsymbol{p},u)) - \beta_1(\boldsymbol{p})\right]} \frac{dw}{(1 + \lambda w^2)} = \beta_2(\boldsymbol{p}) + u. \tag{30}$$

Six cases of Eq. (30) must be put in the form of the proposition: $\lambda > 0$; $\lambda = 0$; and $\lambda < 0$; for each of $f(m) = m^{\kappa}$ and $f(m) = \ln m$.

7.1.1. Extended PIGL

For the extended PIGL[11] and $\lambda > 0$, use,

$$\int_0^x \frac{ds}{(1 + s^2)^{-1}} = \tan^{-1}(x).$$

Let $\lambda = \mu^2 > 0$ and $s = \mu w$, so that Eq. (30) becomes

$$\begin{aligned}\int^{-\beta_3(\boldsymbol{p})/[e(\boldsymbol{p},u)^{\kappa} - \beta_1(\boldsymbol{p})]} \frac{dw}{(1 + \mu^2 w^2)} &= \frac{1}{\mu}\tan^{-1}\left\{\frac{-\mu\beta_3(\boldsymbol{p})}{e(\boldsymbol{p},u)^{\kappa} - \beta_1(\boldsymbol{p})}\right\} \\ &= \beta_2(\boldsymbol{p}) + c(u).\end{aligned} \tag{31}$$

The functions $\beta_1(\boldsymbol{p})$ and $\beta_3(\boldsymbol{p})$ are κ° homogeneous, whereas $\beta_2(\boldsymbol{p})$ is 0° homogeneous. Define $\tilde{\beta}_1(\boldsymbol{p}) \equiv \beta_1(\boldsymbol{p})^{1/\kappa}$ and $\tilde{\beta}_3(\boldsymbol{p}) \equiv \beta_3(\boldsymbol{p})/\beta_1(\boldsymbol{p})$, so that $\tilde{\beta}_1(\boldsymbol{p})$ is 1° homogeneous, whereas $\tilde{\beta}_3(\boldsymbol{p})$ is 0° homogeneous. Apply the normalization $c(u) = \mu^{-1}\tan^{-1}(u)$, the rule for the tangent of the sum of

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[11] Recall that the QES is a special case of the extended PIGL model with $\kappa = 1$.

two angles, $\tan(x+y) = (\tan x + \tan y)/(1 - \tan x \tan y)$, and the identities $\tan(x) = \sin(x)/\cos(x)$, and $\tan(-x) = -\tan(x)$ to rewrite Eq. (31) as

$$\frac{-\mu\tilde{\beta}_3(\boldsymbol{p})}{[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})]^{\kappa} - 1} = \frac{\cos\left[\mu\beta_2(\boldsymbol{p})\right]\cdot u + \sin\left[\mu\beta_2(\boldsymbol{p})\right]}{-\sin\left[\mu\beta_2(\boldsymbol{p})\right]\cdot u + \cos\left[\mu\beta_2(\boldsymbol{p})\right]}. \tag{32}$$

Rearrange terms to obtain

$$\left(\frac{e}{\tilde{\beta}_1}\right)^{\kappa} = \frac{\left[\cos(\mu\beta_2) + \mu\tilde{\beta}_3\sin(\mu\beta_2)\right]\cdot u + \left[\sin(\mu\beta_2) - \mu\tilde{\beta}_3\cos(\mu\beta_2)\right]}{\cos(\mu\beta_2)\cdot u + \sin(\mu\beta_2)}. \tag{33}$$

For these implied definitions of $\{\alpha,\beta,\gamma,\delta\}$, we have $\alpha\delta - \beta\gamma = \mu\tilde{\beta}_3$. Therefore, define $\{\alpha,\beta,\gamma,\delta,\pi\}$ as follows:

$$\begin{aligned}
\pi(\boldsymbol{p}) &= \tilde{\beta}_1(\boldsymbol{p});\\
\alpha(\boldsymbol{p}) &= \frac{\left[\cos(\mu\beta_2(\boldsymbol{p})) + \mu\tilde{\beta}_3(\boldsymbol{p})\sin(\mu\beta_2(\boldsymbol{p}))\right]}{\sqrt{\mu\tilde{\beta}_3(\boldsymbol{p})}};\\
\beta(\boldsymbol{p}) &= \frac{\left[\sin(\mu\beta_2(\boldsymbol{p})) - \mu\tilde{\beta}_3(\boldsymbol{p})\cos(\mu\beta_2(\boldsymbol{p}))\right]}{\sqrt{\mu\tilde{\beta}_3(\boldsymbol{p})}};\\
\gamma(\boldsymbol{p}) &= \frac{\cos(\mu\beta_2(\boldsymbol{p}))}{\sqrt{\mu\tilde{\beta}_3(\boldsymbol{p})}};\\
\delta(\boldsymbol{p}) &= \frac{\sin(\mu\beta_2(\boldsymbol{p}))}{\sqrt{\mu\tilde{\beta}_3(\boldsymbol{p})}}.
\end{aligned}$$

Since $\tilde{\beta}_1$ is 1° homogeneous, while $\beta_2, \tilde{\beta}_3$ are 0° homogeneous, π is 1° homogeneous, $\alpha,\beta,\gamma,\delta$ are 0° homogeneous, $\alpha\delta - \beta\gamma = 1$, and Eq. (33) is equivalent to

$$\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)^{\kappa} = \frac{\alpha(\boldsymbol{p})\cdot u + \beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})\cdot u + \delta(\boldsymbol{p})}. \tag{34}$$

Note that the new definitions for $\{\alpha,\beta,\gamma,\delta\}$ simply rescale these price indices with no change in the indirect preferences or the demand equations. The normalization for the utility index, that is, the arbitrary constant of integration, also can be freely chosen in any way that is most convenient. These properties are exploited as necessary in each of the remaining cases.

For the case where $\lambda = 0$,

$$\int^{-\beta_3(\boldsymbol{p})/[e(\boldsymbol{p},u)^{\kappa} - \beta_1(\boldsymbol{p})]} dw = \frac{-\beta_3(\boldsymbol{p})}{e(\boldsymbol{p},u)^{\kappa} - \beta_1(\boldsymbol{p})} = \beta_2(\boldsymbol{p}) + c(u).$$

Define $\tilde{\beta}_1(\boldsymbol{p})$ and $\tilde{\beta}_3(\boldsymbol{p})$ in the same way as aforementioned, apply the normalization $c(u) = u$, and rearrange terms to obtain,

$$\left(\frac{e(\boldsymbol{p}, u)}{\tilde{\beta}_1(\boldsymbol{p})}\right)^{\kappa} = \frac{u + \beta_2(\boldsymbol{p}) - \tilde{\beta}_3(\boldsymbol{p})}{u + \beta_2(\boldsymbol{p})}.$$

For these implied definitions of $\{\alpha, \beta, \gamma, \delta\}$, that is, $\alpha = \gamma = 1$, $\beta = \beta_2 - \tilde{\beta}_3$, and $\delta = \beta_2$, we have $\alpha\delta - \beta\gamma = \tilde{\beta}_3$. Therefore, define $\{\alpha, \beta, \gamma, \delta, \pi\}$ as follows:

$$\pi(\boldsymbol{p}) = \tilde{\beta}_1(\boldsymbol{p});$$

$$\alpha(\boldsymbol{p}) = \gamma(\boldsymbol{p}) = 1 \Big/ \sqrt{\tilde{\beta}_3(\boldsymbol{p})};$$

$$\beta(\boldsymbol{p}) = \frac{\left[\beta_2(\boldsymbol{p}) - \tilde{\beta}_3(\boldsymbol{p})\right]}{\sqrt{\tilde{\beta}_3(\boldsymbol{p})}};$$

$$\delta(\boldsymbol{p}) = \frac{\beta_2(\boldsymbol{p})}{\sqrt{\tilde{\beta}_3(\boldsymbol{p})}}.$$

Then we again obtain Eq. (34).

Next, let $\lambda = -\mu^2 < 0$ in Eq. (30), so that

$$\int^{-\beta_3(\boldsymbol{p})/[e(\boldsymbol{p},u)^{\kappa} - \beta_1(\boldsymbol{p})]} \frac{dw}{(1 - \mu^2 w^2)} = \frac{1}{2\mu} \ln\left\{\frac{e(\boldsymbol{p}, u)^{\kappa} - \beta_1(\boldsymbol{p}) - \mu\beta_3(\boldsymbol{p})}{e(\boldsymbol{p}, u)^{\kappa} - \beta_1(\boldsymbol{p}) + \mu\beta_3(\boldsymbol{p})}\right\} = \beta_2(\boldsymbol{p}) + c(u).$$

Define $\tilde{\beta}_1(\boldsymbol{p})$ and $\tilde{\beta}_3(\boldsymbol{p})$ in the same way as in the previous two cases and apply the normalization $c(u) = \ln(u)/2\mu$ to rewrite this as

$$\frac{\left[e(\boldsymbol{p}, u)/\tilde{\beta}_1(\boldsymbol{p})\right]^{\kappa} - 1 - \mu\tilde{\beta}_3(\boldsymbol{p})}{\left[e(\boldsymbol{p}, u)/\tilde{\beta}_1(\boldsymbol{p})\right]^{\kappa} - 1 + \mu\tilde{\beta}_3(\boldsymbol{p})} = \mathrm{e}^{2\mu\beta_2(\boldsymbol{p})} \cdot u.$$

Rearranging terms yields

$$\left(\frac{e(\boldsymbol{p}, u)}{\tilde{\beta}_1(\boldsymbol{p})}\right)^{\kappa} = \frac{[1 - \mu\tilde{\beta}_3(\boldsymbol{p})]\mathrm{e}^{2\mu\beta_2(\boldsymbol{p})} \cdot u - [1 + \mu\tilde{\beta}_3(\boldsymbol{p})]}{\mathrm{e}^{2\mu\beta_2(\boldsymbol{p})} \cdot u - 1}. \tag{35}$$

For these implied definitions of $\{\alpha, \beta, \gamma, \delta\}$, we have $\alpha\delta - \beta\gamma = 2\mu\tilde{\beta}_3\mathrm{e}^{2\mu\beta_2}$. Therefore, define $\{\alpha, \beta, \gamma, \delta, \pi\}$ as follows:

$$\pi(\boldsymbol{p}) = \tilde{\beta}_1(\boldsymbol{p});$$

$$\alpha(\boldsymbol{p}) = \frac{[1 - \mu\tilde{\beta}_3(\boldsymbol{p})]\mathrm{e}^{\mu\beta_2(\boldsymbol{p})}}{\sqrt{2\mu\tilde{\beta}_3(\boldsymbol{p})}};$$

$$\beta(\boldsymbol{p}) = \frac{-[1 + \mu\tilde{\beta}_3(\boldsymbol{p})]\mathrm{e}^{-\mu\tilde{\beta}_2(\boldsymbol{p})}}{\sqrt{2\mu\tilde{\beta}_3(\boldsymbol{p})}};$$

$$\gamma(\boldsymbol{p}) = \frac{e^{\mu\beta_2(\boldsymbol{p})}}{\sqrt{2\mu\tilde{\beta}_3(\boldsymbol{p})}};$$

$$\delta(\boldsymbol{p}) = -\frac{e^{-\mu\tilde{\beta}_2(\boldsymbol{p})}}{\sqrt{2\mu\tilde{\beta}_3(\boldsymbol{p})}}.$$

Then we once again obtain Eq. (34).

7.1.2. Extended PIGLOG

The same three cases for λ apply to the extended PIGLOG, except that $\ln m$ replaces m^κ everywhere, $\beta_1(\boldsymbol{p}) = \ln \tilde{\beta}_1(\boldsymbol{p})$ with $\tilde{\beta}_1(\boldsymbol{p})$ 1° homogeneous, and both $\beta_2(\boldsymbol{p})$ and $\beta_3(\boldsymbol{p})$ are 0° homogeneous.

When $\lambda = \mu^2 > 0$, Eq. (31) becomes

$$\int^{-\beta_3(\boldsymbol{p})/\ln[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})]} \frac{dw}{(1+\mu^2 w^2)} = \frac{1}{\mu}\tan^{-1}\left\{\frac{-\mu\beta_3(\boldsymbol{p})}{\ln[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})]}\right\}$$
$$= \beta_2(\boldsymbol{p}) + c(u).$$

Applying the same trigonometric rules and the normalization $c(u) = \mu^{-1}\tan^{-1}(u)$, this can be rewritten as

$$\frac{-\mu\beta_3(\boldsymbol{p})}{\ln[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})]} = \frac{\cos[\mu\beta_2(\boldsymbol{p})]\cdot u + \sin[\mu\beta_2(\boldsymbol{p})]}{-\sin[\mu\beta_2(\boldsymbol{p})]\cdot u + \cos[\mu\beta_2(\boldsymbol{p})]},$$

Rearranging terms yields

$$\ln\left(\frac{e(\boldsymbol{p},u)}{\tilde{\beta}_1(\boldsymbol{p})}\right) = \mu\beta_3(\boldsymbol{p})\left(\frac{\sin[\mu\beta_2(\boldsymbol{p})]\cdot u - \cos[\mu\beta_2(\boldsymbol{p})]}{\cos[\mu\beta_2(\boldsymbol{p})]\cdot u + \sin[\mu\beta_2(\boldsymbol{p})]}\right).$$

For these implicit definitions for $\{\alpha,\beta,\gamma,\delta\}$, we have $\alpha\delta - \beta\gamma = \mu\beta_3$. Therefore, define $\{\alpha,\beta,\gamma,\delta,\pi\}$ as follows:

$$\pi(\boldsymbol{p}) = \tilde{\beta}_1(\boldsymbol{p});$$

$$\alpha(\boldsymbol{p}) = \sqrt{\mu\beta_3(\boldsymbol{p})}\sin(\mu\beta_2(\boldsymbol{p}));$$

$$\beta(\boldsymbol{p}) = -\sqrt{\mu\tilde{\beta}_3(\boldsymbol{p})}\cos(\mu\beta_2(\boldsymbol{p}));$$

$$\gamma(\boldsymbol{p}) = \frac{\cos(\mu\beta_2(\boldsymbol{p}))}{\sqrt{\mu\tilde{\beta}_3(\boldsymbol{p})}};$$

$$\delta(\boldsymbol{p}) = \frac{\sin(\mu\beta_2(\boldsymbol{p}))}{\sqrt{\mu\tilde{\beta}_3(\boldsymbol{p})}}.$$

Then we have

$$\ln\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)=\frac{\alpha(\boldsymbol{p})\cdot u+\beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})\cdot u+\delta(\boldsymbol{p})}, \tag{36}$$

with π 1° homogeneous, $\alpha,\beta,\gamma,\delta$ 0° homogeneous, and $\alpha\delta-\beta\gamma=1$.

Similarly, if $\lambda=0$, then

$$\int^{-\beta_3(\boldsymbol{p})/\ln[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})]} dw=\frac{-\beta_3(\boldsymbol{p})}{\ln[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})]}=\beta_2(\boldsymbol{p})+c(u).$$

Apply the normalization $c(u)=u$ and rearrange terms to obtain

$$\ln\left(\frac{e(\boldsymbol{p},u)}{\tilde{\beta}_1(\boldsymbol{p})}\right)=\frac{-\beta_3(\boldsymbol{p})}{u+\beta_2(\boldsymbol{p})}.$$

For these definitions for $\{\alpha,\beta,\gamma,\delta\}$, that is, $\alpha=0,\ \beta=-\beta_3,\ \gamma=1$, and $\delta=\beta_2$, we have $\alpha\delta-\beta\gamma=\beta_3$. Therefore, define $\{\alpha,\beta,\gamma,\delta,\pi\}$ as follows:

$$\begin{aligned}
\pi(\boldsymbol{p})&=\tilde{\beta}_1(\boldsymbol{p});\\
\alpha(\boldsymbol{p})&=0;\\
\beta(\boldsymbol{p})&=-\sqrt{\beta_3(\boldsymbol{p})};\\
\gamma(\boldsymbol{p})&=\frac{1}{\sqrt{\beta_3(\boldsymbol{p})}};\\
\delta(\boldsymbol{p})&=\frac{\beta_2(\boldsymbol{p})}{\sqrt{\beta_3(\boldsymbol{p})}}.
\end{aligned}$$

Then we have Eq. (36), π 1° homogeneous, $\alpha,\beta,\gamma,\delta$ 0° homogeneous, and $\alpha\delta-\beta\gamma=1$.

Finally, if $\lambda=-\mu^2<0$, then

$$\begin{aligned}
\int^{-\beta_3(\boldsymbol{p})/\ln[e(\boldsymbol{p},u)/\tilde{\beta}_1(\boldsymbol{p})]}\frac{dw}{(1-\mu^2w^2)}&=\frac{1}{2\mu}\ln\left\{\frac{\ln[e(\boldsymbol{p},u)/\beta_1(\boldsymbol{p})]-\mu\beta_3(\boldsymbol{p})}{\ln[e(\boldsymbol{p},u)/\beta_1(\boldsymbol{p})]+\mu\beta_3(\boldsymbol{p})}\right\}\\
&=\beta_2(\boldsymbol{p})+c(u).
\end{aligned}$$

Apply the normalization $c(u)=\ln(u)/2\mu$ to rewrite this as

$$\frac{\ln[e(\boldsymbol{p},u)/\beta_1(\boldsymbol{p})]-\mu\beta_3(\boldsymbol{p})}{\ln[e(\boldsymbol{p},u)/\beta_1(\boldsymbol{p})]+\mu\beta_3(\boldsymbol{p})}=\mathrm{e}^{\beta_2(\boldsymbol{p})}\cdot u.$$

Rearranging terms, this is equivalent to

$$\ln\left(\frac{e(\boldsymbol{p},u)}{\tilde{\beta}_1(\boldsymbol{p})}\right)=\mu\beta_3(\boldsymbol{p})\left(\frac{\mathrm{e}^{\beta_2(\boldsymbol{p})}\cdot u+1}{-\mathrm{e}^{\beta_2(\boldsymbol{p})}\cdot u+1}\right).$$

For these implied definitions for $\{\alpha, \beta, \gamma, \delta\}$, we have $\alpha\delta - \beta\gamma = 2\mu\beta_3 e^{\beta_2}$. Hence, define $\{\alpha, \beta, \gamma, \delta, \pi\}$ as follows:

$$\pi(\boldsymbol{p}) = \tilde{\beta}_1(\boldsymbol{p});$$

$$\alpha(\boldsymbol{p}) = \sqrt{\frac{\mu\beta_3(\boldsymbol{p})}{2}} \cdot e^{\frac{1}{2}\beta_2(\boldsymbol{p})};$$

$$\beta(\boldsymbol{p}) = \sqrt{\frac{\mu\beta_3(\boldsymbol{p})}{2}} \cdot e^{-\frac{1}{2}\beta_2(\boldsymbol{p})};$$

$$\gamma(\boldsymbol{p}) = -\frac{e^{\frac{1}{2}\beta_2(\boldsymbol{p})}}{\sqrt{2\mu\beta_3(\boldsymbol{p})}};$$

$$\delta(\boldsymbol{p}) = \frac{e^{-\frac{1}{2}\beta_2(\boldsymbol{p})}}{\sqrt{2\mu\beta_3(\boldsymbol{p})}}.$$

Then we have Eq. (36), π 1° homogeneous, $\alpha, \beta, \gamma, \delta$ 0° homogeneous, and $\alpha\delta - \beta\gamma = 1$.

This completes the proof of necessity for the extended PIGL and PIGLOG models for a constant $\theta(x) \equiv \lambda$. For the extended PIGL and PIGLOG models with $\theta'(\beta_3(\boldsymbol{p})) \neq 0$, write

$$w(\beta_3(\boldsymbol{p}), u) = \frac{f(e(\boldsymbol{p}, u)) - \beta_1(\boldsymbol{p})}{\beta_2(\boldsymbol{p})}, \tag{37}$$

with $w(\beta_3(\boldsymbol{p}), u)$ defined in the second line of Eq. (26) and $\beta_3(\boldsymbol{p}) = \eta(\boldsymbol{p})$. If $f(m) = m^\kappa$, then rewrite Eq. (37) as

$$\left(\frac{e(\boldsymbol{p}, u)}{\tilde{\beta}_2(\boldsymbol{p})}\right)^\kappa = w(\beta_3(\boldsymbol{p}), u) + \tilde{\beta}_1(\boldsymbol{p}),$$

where $\tilde{\beta}_2 = \beta_2^{1/\kappa}$ is 1° homogeneous and $\tilde{\beta}_1 = \beta_1/\beta_2$ is 0° homogeneous. The implied definitions of $\{\alpha, \beta, \gamma, \delta, \pi\}$ are $\alpha = \delta = 1$, $\beta = -\tilde{\beta}_1$, $\gamma = 0$, and $\pi = \tilde{\beta}_2$.

Similarly, if $f(m) = \ln m$, then rewrite Eq. (37) as

$$\ln\left(\frac{e(\boldsymbol{p}, u)}{\tilde{\beta}_1(\boldsymbol{p})}\right) = \beta_2(\boldsymbol{p}) \cdot w(\beta_3(\boldsymbol{p}), u),$$

where $\beta_1 = \ln \tilde{\beta}_1$ and $\tilde{\beta}_1$ is 1° homogeneous. The definitions of $\{\alpha, \beta, \gamma, \delta, \pi\}$ that lead to the representation given in the proposition are $\alpha = \sqrt{\beta_2}$, $\beta = \gamma = 0$, $\delta = 1/\sqrt{\beta_2}$, and $\pi = \tilde{\beta}_1$. This completes the proof of necessity for all full rank three extended PIGL or PIGLOG demand systems.

7.1.3. Trigonometric

The only remaining case for a full rank three Gorman system is the trigonometric indirect utility function found by Lewbel (1988, 1990),

$$v(\boldsymbol{p},m) = \beta_2(\boldsymbol{p}) + \frac{\beta_3(\boldsymbol{p})\cos\left[\tau \ln(m/\beta_1(\boldsymbol{p}))\right]}{\left[1-\sin\left[\tau \ln(m/\beta_1(\boldsymbol{p}))\right]\right]}, \tag{38}$$

with β_1 1° homogeneous and β_2, β_3 0° homogeneous. Apply the definitions of and rules for calculating sums and differences of sine and cosine functions (e.g., Abramowitz and Stegun, 1972, pp. 71–74), to rewrite Eq. (38) as

$$v(\boldsymbol{p},m) = \frac{\left[\beta_3(\boldsymbol{p}) - \imath\beta_2(\boldsymbol{p})\right] \times [m/\beta_1(\boldsymbol{p})]^{\imath\tau} + \beta_2(\boldsymbol{p}) - \imath\beta_3(\boldsymbol{p})}{1-\imath[m/\beta_1(\boldsymbol{p})]^{\imath\tau}}. \tag{39}$$

To obtain the representation in the proposition, appropriate transformations of income and the price indices must be found. Set $v(\boldsymbol{p},m) = u$ and $m = e(\boldsymbol{p},u)$ and invert Eq. (39) to yield,

$$\left(\frac{e(\boldsymbol{p},u)}{\beta_1(\boldsymbol{p})}\right)^{\imath\tau} = \frac{u - \beta_2(\boldsymbol{p}) + \imath\cdot\beta_3(\boldsymbol{p})}{\imath\cdot u + \beta_3(\boldsymbol{p}) - \imath\cdot\beta_2(\boldsymbol{p})}.$$

For the implied definitions of $\{\alpha,\beta,\gamma,\delta\}$, we have $\alpha\delta - \beta\gamma = 2\beta_3(\boldsymbol{p})$. Therefore, define $\{\alpha,\beta,\gamma,\delta,\pi\}$ as follows:

$$\begin{aligned}
\pi(\boldsymbol{p}) &= \beta_1(\boldsymbol{p});\\
\alpha(\boldsymbol{p}) &= \frac{1}{\sqrt{2\beta_3(\boldsymbol{p})}};\\
\beta(\boldsymbol{p}) &= \frac{\left[-\beta_2(\boldsymbol{p}) + \imath\cdot\beta_3(\boldsymbol{p})\right]}{\sqrt{2\beta_3(\boldsymbol{p})}};\\
\gamma(\boldsymbol{p}) &= \frac{\imath}{\sqrt{2\beta_3(\boldsymbol{p})}};\\
\delta(\boldsymbol{p}) &= \frac{\left[\beta_3(\boldsymbol{p}) - \imath\cdot\beta_2(\boldsymbol{p})\right]}{\sqrt{2\beta_3(\boldsymbol{p})}}.
\end{aligned}$$

This yields

$$\left(\frac{e(\boldsymbol{p},u)}{\pi(\boldsymbol{p})}\right)^{\imath\tau} = \frac{\alpha(\boldsymbol{p})\cdot u + \beta(\boldsymbol{p})}{\gamma(\boldsymbol{p})\cdot u + \delta(\boldsymbol{p})},$$

with π 1° homogeneous, $\alpha,\beta,\gamma,\delta$ 0° homogeneous, and $\alpha\delta - \beta\gamma = 1$, as required.

Thus, all full rank Gorman systems can be written in the form given in the proposition. It is worth emphasizing that in each case, $\{\alpha,\beta,\gamma,\delta,\eta\}$ depend on *at most two linearly independent price indices*. It also is important to note that the Gorman functional forms are responsible for the property that one 1° homogeneous price index can be extracted to deflate income. This is the fundamental role of the Gorman functional forms in a nominal income Gorman system.

Now turn to the proof of necessity for all Lewbel systems.

7.2. Lewbel systems

Recall Eq. (12),

$$\frac{\partial \tilde{e}}{\partial \boldsymbol{p}} = \sum_{k=2}^{K} \tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p}) h_k(\tilde{e}).$$

$K = 1$ repeats the homothetic, full rank one case, and does not require additional proof. If $K \geq 2$, then linear independence of the $\{h_2, \ldots, h_K\}$ implies that at least one of these functions cannot vanish. WLOG, let it be h_2 and define the map $y = f(\tilde{e})$ by

$$y(\boldsymbol{p}, u) = f(\tilde{e}(\boldsymbol{p}, u)) = \int^{\tilde{e}(\boldsymbol{p}, u)} \frac{dx}{h_2(x)}.$$

Then by Leibnitz' rule, we have

$$\begin{aligned}
\frac{\partial y(\boldsymbol{p}, u)}{\partial \boldsymbol{p}} &= \frac{1}{h_2(\tilde{e}(\boldsymbol{p}, u))} \times \frac{1}{\pi(\boldsymbol{p})} \left[\boldsymbol{q} - \tilde{e}(\boldsymbol{p}, u) \frac{\partial \pi(\boldsymbol{p})}{\partial \boldsymbol{p}} \right] \\
&= \tilde{\boldsymbol{\alpha}}_2(\boldsymbol{p}) + \sum_{k=3}^{K} \tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p}) \frac{h_k(\tilde{e}(\boldsymbol{p}, u))}{h_2(\tilde{e}(\boldsymbol{p}, u))} \\
&= \tilde{\boldsymbol{\alpha}}_2(\boldsymbol{p}) + \sum_{k=3}^{K} \tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p}) \tilde{h}_k(\tilde{e}(\boldsymbol{p}, u)),
\end{aligned}$$

where $\tilde{h}_k(\tilde{e}(\boldsymbol{p}, u)) = h_k(\tilde{e}(\boldsymbol{p}, u))/h_2(\tilde{e}(\boldsymbol{p}, u)),\ k = 3, \ldots, K.$ Since $h_2(x) \neq 0$, $f^{-1}(y)$ exists, so that

$$\partial y(\boldsymbol{p}, u)/\partial \boldsymbol{p} = \tilde{\boldsymbol{\alpha}}_2(\boldsymbol{p}) + \sum_{k=3}^{4} \tilde{\boldsymbol{\alpha}}_k(\boldsymbol{p}) \hat{h}_k(y(\boldsymbol{p}, u)), \tag{40}$$

where $\hat{h}_k(y(\boldsymbol{p}, u)) \equiv \tilde{h}_k(f^{-1}(y(\boldsymbol{p}, u))),\ \hat{h}_k : \mathbb{R} \to \mathbb{R},\ \hat{h}_k \in \mathcal{C}^\infty,\ k = 3, \ldots, K.$ These steps reduce the demand system to one in which the first income term on the right-hand side is the constant function, that is, $\tilde{h}_2(y) \equiv 1$, maintaining the additive structure of Gorman, but now with multiplicative separability between $\boldsymbol{p}$ and y, rather than $\boldsymbol{p}$ and $\tilde{e}$.

From the results of Lewbel (1989a) and Lie (Hermann, 1975), we know that $K \leq 4$ in any full Lewbel rank system. Hence, all solutions to Eq. (40) for $K = 2, 3, 4$ must be found. To simplify the notational burden, drop all of the $\sim$s and $\hat{\text{s}}$ and rewrite Eq. (40) as

$$\frac{\partial y(\boldsymbol{p}, u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2(\boldsymbol{p}) + \sum_{k=3}^{K} \boldsymbol{\alpha}_k(\boldsymbol{p}) h_k(y(\boldsymbol{p}, u)). \tag{41}$$

$$K = 2: \quad \partial y(\boldsymbol{p}, u)/\partial \boldsymbol{p} = \boldsymbol{\alpha}_2(\boldsymbol{p}).$$

This implies $\partial^2 y/\partial \boldsymbol{p}\partial \boldsymbol{p}^\top = \partial \boldsymbol{\alpha}_2/\partial \boldsymbol{p}^\top$, so that $\partial \boldsymbol{\alpha}_2/\partial \boldsymbol{p}^\top$ is symmetric. This is necessary and sufficient for the existence of a 0° homogeneous function, $\beta : \mathbb{R}^n_+ \to \mathbb{R}$, $\beta \in \mathcal{C}^\infty$, such that $\partial \beta(\boldsymbol{p})/\partial \boldsymbol{p} = \boldsymbol{\alpha}_2(\boldsymbol{p})$. Integrating Eq. (41) then yields

$$y(\boldsymbol{p}, u) = u + \beta(\boldsymbol{p}),$$

with an obvious normalization for u. The implied definitions $\alpha = \delta = 1$ and $\gamma = 0$ yield the representation given in the proposition.

$$K = 3: \quad \frac{\partial y(\boldsymbol{p}, u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2(\boldsymbol{p}) + \boldsymbol{\alpha}_3(\boldsymbol{p}) h_3(y(\boldsymbol{p}, u)).$$

This implies

$$\begin{aligned}\frac{\partial^2 y}{\partial \boldsymbol{p}\partial \boldsymbol{p}^\top} &= \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{p}^\top} + \frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{p}^\top} h_3 + \boldsymbol{\alpha}_3 \boldsymbol{\alpha}_2^\top h_3' + \boldsymbol{\alpha}_3 \boldsymbol{\alpha}_3^\top h_3 h_3' \\ &= \frac{\partial \boldsymbol{\alpha}_2^\top}{\partial \boldsymbol{p}} + \frac{\partial \boldsymbol{\alpha}_3^\top}{\partial \boldsymbol{p}} h_3 + \boldsymbol{\alpha}_2 \boldsymbol{\alpha}_3^\top h_3' + \boldsymbol{\alpha}_3 \boldsymbol{\alpha}_3^\top h_3 h_3'.\end{aligned}$$

Subtracting the far right expression from the middle one implies,

$$(\boldsymbol{\alpha}_3 \boldsymbol{\alpha}_2^\top - \boldsymbol{\alpha}_2 \boldsymbol{\alpha}_3^\top) h_3' = \left(\frac{\partial \boldsymbol{\alpha}_2^\top}{\partial \boldsymbol{p}} - \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{p}^\top}\right) + \left(\frac{\partial \boldsymbol{\alpha}_3^\top}{\partial \boldsymbol{p}} - \frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{p}^\top}\right) h_3. \tag{42}$$

Since $\{\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$ are linearly independent, $\boldsymbol{\alpha}_3 \neq c\boldsymbol{\alpha}_2$ for any $c \in \mathbb{R}$. Hence, $\boldsymbol{\alpha}_3 \boldsymbol{\alpha}_2^\top$ is not symmetric. Since $\{1, h_3(y)\}$ are linearly independent, $h_3' \neq 0$. Premultiply Eq. (42) by $\boldsymbol{\alpha}_3^\top$, postmultiply by $\boldsymbol{\alpha}_2$, and divide by $\boldsymbol{\alpha}_3^\top \boldsymbol{\alpha}_3 \boldsymbol{\alpha}_2^\top \boldsymbol{\alpha}_2 - (\boldsymbol{\alpha}_3^\top \boldsymbol{\alpha}_2)^2 > 0$ (by the Cauchy–Schwartz inequality) to obtain

$$h_3'(y) = c_1 + c_2 h_3(y), \tag{43}$$

where c_1 and c_2 are absolute constants since $h_3(y)$ and $h_3'(y)$ are independent of $\boldsymbol{p}$. In other words, the solution to this differential equation, which is $h_3(y)$ by definition, is not a function of prices.

If $c_2 \neq 0$, the solution to this linear, first-order, ode is $h_3(y) = -(c_1/c_2) + c_3 e^{c_2 y}$, where c_3 is a constant of integration. Plugging this into Eq. (42) then implies that the $n \times n$ matrix equation,

$$(\boldsymbol{\alpha}_3 \boldsymbol{\alpha}_2^\top - \boldsymbol{\alpha}_2 \boldsymbol{\alpha}_3^\top) c_2 c_3 e^{c_2 y} = \left(\frac{\partial \boldsymbol{\alpha}_2^\top}{\partial \boldsymbol{p}} - \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{p}^\top}\right) + \left(\frac{\partial \boldsymbol{\alpha}_3^\top}{\partial \boldsymbol{p}} - \frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{p}^\top}\right)\left[-\left(\frac{c_1}{c_2}\right) + c_3 e^{c_2 y}\right],$$

holds identically in $(\boldsymbol{p}, y)$. But this implies that $c_3 = 0$, which contradicts the linear independence of $\{1, h_3(y)\} = \{1, -(c_1/c_2) + c_3 e^{c_2 y}\}$.

Therefore, it must be that $c_2 = 0$ and the complete solution to Eq. (43) is $h_3(y) = c_1 y + b$ for some constant of integration b. WLOG, absorb the constants c_1 and b into $\boldsymbol{\alpha}_2(\boldsymbol{p})$ and $\boldsymbol{\alpha}_3(\boldsymbol{p})$ by linear transformations, tacitly normalizing so that $h_2(y) = 1$ and $h_3(y) = y$, which are linearly independent. The system of demand equations then is

$$\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2(\boldsymbol{p}) + \boldsymbol{\alpha}_3(\boldsymbol{p})y(\boldsymbol{p},u). \tag{44}$$

As a result, symmetry reduces to

$$\frac{\partial^2 y}{\partial \boldsymbol{p}\partial \boldsymbol{p}^\top} = \frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{p}^\top} + \boldsymbol{\alpha}_3\boldsymbol{\alpha}_2^\top + \left(\frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{p}^\top} + \boldsymbol{\alpha}_3\boldsymbol{\alpha}_3^\top\right)y = \frac{\partial \boldsymbol{\alpha}_2^\top}{\partial \boldsymbol{p}} + \boldsymbol{\alpha}_2\boldsymbol{\alpha}_3^\top + \left(\frac{\partial \boldsymbol{\alpha}_3^\top}{\partial \boldsymbol{p}} + \boldsymbol{\alpha}_3\boldsymbol{\alpha}_3^\top\right)y.$$

Equating like powers in y, $\partial\boldsymbol{\alpha}_3/\partial\boldsymbol{p}^\top$ is symmetric. This is necessary and sufficient for a 0° homogeneous function, $\varphi : \mathbb{R}^n_+ \to \mathbb{R}$, to exist such that $\partial\varphi(\boldsymbol{p})/\partial\boldsymbol{p} = \boldsymbol{\alpha}_3(\boldsymbol{p})$. Substituting this into Eq. (44) yields

$$\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2(\boldsymbol{p}) + \frac{\partial\varphi(\boldsymbol{p})}{\partial \boldsymbol{p}}y(\boldsymbol{p},u) \tag{45}$$

Symmetry now reduces to

$$\frac{\partial \boldsymbol{\alpha}_2}{\partial \boldsymbol{p}^\top} + \frac{\partial\varphi}{\partial \boldsymbol{p}}\boldsymbol{\alpha}_2^\top = \frac{\partial \boldsymbol{\alpha}_2^\top}{\partial \boldsymbol{p}} + \boldsymbol{\alpha}_2\frac{\partial\varphi}{\partial \boldsymbol{p}^\top},$$

which implies that $\partial\boldsymbol{\alpha}_2/\partial\boldsymbol{p}^\top - \boldsymbol{\alpha}_2\,\partial\varphi/\partial\boldsymbol{p}^\top$is symmetric.

Therefore, applying the integrating factor $e^{-\varphi}$ to Eq. (45) yields

$$\frac{\partial}{\partial \boldsymbol{p}}\left[y(\boldsymbol{p},u)e^{-\varphi(\boldsymbol{p})}\right] = \left[\frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} - y(\boldsymbol{p},u)\frac{\partial\varphi(\boldsymbol{p})}{\partial \boldsymbol{p}}\right]e^{-\varphi(\boldsymbol{p})} = \boldsymbol{\alpha}_2(\boldsymbol{p})e^{-\varphi(\boldsymbol{p})}.$$

Differentiating this with respect to $\boldsymbol{p}^\top$ then implies

$$\frac{\partial^2}{\partial \boldsymbol{p}\partial \boldsymbol{p}^\top}\left[y(\boldsymbol{p},u)e^{-\varphi(\boldsymbol{p})}\right] = \left[\frac{\partial \boldsymbol{\alpha}_2(\boldsymbol{p})}{\partial \boldsymbol{p}^\top} - \boldsymbol{\alpha}_2(\boldsymbol{p})\frac{\partial\varphi(\boldsymbol{p})}{\partial \boldsymbol{p}^\top}\right]e^{-\varphi(\boldsymbol{p})}.$$

Symmetry of the $n \times n$ matrix on the right-hand side implies that a 0° homogeneous function, $\boldsymbol{\rho} : \mathcal{P} \to \mathbb{R}$, exists such that $\partial\rho(\boldsymbol{p})/\partial\boldsymbol{p} = \boldsymbol{\alpha}_2(\boldsymbol{p})e^{-\varphi(\boldsymbol{p})}$ and

$$y(\boldsymbol{p},u)e^{-\varphi(\boldsymbol{p})} = u + \rho(\boldsymbol{p}),$$

with an obvious normalization for u. Solve this for $y(\boldsymbol{p}, u)$ and define $\alpha(\boldsymbol{p}) = e^{\frac{1}{2}\varphi(\boldsymbol{p})}$, $\beta(\boldsymbol{p}) = e^{\frac{1}{2}\varphi(\boldsymbol{p})}\rho(\boldsymbol{p})$, $\gamma(\boldsymbol{p}) = 0$, and $\delta(\boldsymbol{p}) = e^{-\frac{1}{2}\varphi(\boldsymbol{p})}$ to obtain the representation given by the proposition.

$$K = 4: \quad \frac{\partial y(\boldsymbol{p},u)}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2(\boldsymbol{p}) + \boldsymbol{\alpha}_3(\boldsymbol{p})h_3(y(\boldsymbol{p},u)) + \boldsymbol{\alpha}_4(\boldsymbol{p})h_4(y(\boldsymbol{p},u)).$$

We have

$$\begin{aligned}\frac{\partial^2 y}{\partial p_i\partial p_j} &= \frac{\partial\alpha_{i2}}{\partial p_j} + \sum_{k=3}^{4}\frac{\partial\alpha_{ik}}{\partial p_j}h_k + \sum_{k=3}^{4}\alpha_{ik}h'_k\left(\alpha_{j2} + \sum_{\ell=3}^{4}\alpha_{j\ell}h_\ell\right)\\ &= \frac{\partial\alpha_{j2}}{\partial p_i} + \sum_{k=3}^{4}\frac{\partial\alpha_{jk}}{\partial p_i}h_k + \sum_{k=3}^{4}\alpha_{jk}h'_k\left(\alpha_{i2} + \sum_{\ell=3}^{4}\alpha_{i\ell}h_\ell\right) = \frac{\partial^2 y}{\partial p_j\partial p_i}, \;\; \forall\ i\neq j.\end{aligned}$$

Rewrite this in terms of $\frac{1}{2}n(n–1)$ vanishing differences,

$$
\begin{aligned}
0 = & \frac{\partial \alpha_{i2}}{\partial p_j} - \frac{\partial \alpha_{j2}}{\partial p_i} + \left(\frac{\partial \alpha_{i3}}{\partial p_j} - \frac{\partial \alpha_{j3}}{\partial p_i} \right) h_3 + \left(\frac{\partial \alpha_{i4}}{\partial p_j} - \frac{\partial \alpha_{j4}}{\partial p_i} \right) h_4 \\
& + (\alpha_{i3}\alpha_{j2} - \alpha_{i2}\alpha_{j3}) h_3' + (\alpha_{i4}\alpha_{j2} - \alpha_{i2}\alpha_{j4}) h_4' \\
& + \sum_{k=3}^{4} \sum_{\ell=3}^{4} \alpha_{ik}\alpha_{j\ell}(h_k' h_\ell - h_k h_\ell'), \ \ \forall \ j < i = 2, \ldots, n.
\end{aligned}
$$

If $k = \ell$ in the double sum, then $\alpha_{ik}\alpha_{jk}$ is multiplied by $h_k' h_k - h_k h_k' = 0$, whereas if $k \neq \ell$, then $h_k' h_\ell - h_k h_\ell'$ is multiplied once by $\alpha_{ik}\alpha_{j\ell}$ and $-\alpha_{i\ell}\alpha_{jk}$. Thus,

$$
\begin{aligned}
0 = & \frac{\partial \alpha_{i2}}{\partial p_j} - \frac{\partial \alpha_{j2}}{\partial p_i} + \left(\frac{\partial \alpha_{i3}}{\partial p_j} - \frac{\partial \alpha_{j3}}{\partial p_i} \right) h_3 \\
& + \left(\frac{\partial \alpha_{i4}}{\partial p_j} - \frac{\partial \alpha_{j4}}{\partial p_i} \right) h_4 + (\alpha_{i3}\alpha_{j2} - \alpha_{i2}\alpha_{j3}) h_3' + (\alpha_{i4}\alpha_{j2} - \alpha_{i2}\alpha_{j4}) h_4' \\
& + (\alpha_{i4}\alpha_{j3} - \alpha_{i3}\alpha_{j4})(h_3' h_4 - h_3 h_4'), \ \ \forall \ j < i = 2, \ldots, n.
\end{aligned}
$$

Now define the matrices

$$
\boldsymbol{B} = \begin{bmatrix}
\alpha_{23}\alpha_{12} - \alpha_{22}\alpha_{13} & \alpha_{24}\alpha_{12} - \alpha_{22}\alpha_{14} & \alpha_{24}\alpha_{13} - \alpha_{23}\alpha_{14} \\
\alpha_{33}\alpha_{12} - \alpha_{32}\alpha_{13} & \alpha_{34}\alpha_{12} - \alpha_{32}\alpha_{14} & \alpha_{34}\alpha_{13} - \alpha_{33}\alpha_{14} \\
\vdots & \vdots & \vdots \\
\alpha_{n,3}\alpha_{n-1,2} - \alpha_{n,2}\alpha_{n-1,3} & \alpha_{n,4}\alpha_{n-1,2} - \alpha_{n,2}\alpha_{n-13} & \alpha_{n,4}\alpha_{n-1,3} - \alpha_{n-1,3}\alpha_{n,4}
\end{bmatrix},
$$

$$
\boldsymbol{C} = \begin{bmatrix}
\frac{\partial \alpha_{22}}{\partial p_2} - \frac{\partial \alpha_{12}}{\partial p_1} & \frac{\partial \alpha_{23}}{\partial p_2} - \frac{\partial \alpha_{13}}{\partial p_1} & \frac{\partial \alpha_{24}}{\partial p_2} - \frac{\partial \alpha_{14}}{\partial p_1} \\
\vdots & \vdots & \vdots \\
\frac{\partial \alpha_{n,2}}{\partial p_{n-1}} - \frac{\partial \alpha_{n-1,2}}{\partial p_n} & \frac{\partial \alpha_{n,3}}{\partial p_{n-1}} - \frac{\partial \alpha_{n-1,3}}{\partial p_n} & \frac{\partial \alpha_{n,4}}{\partial p_{n-1}} - \frac{\partial \alpha_{n-1,4}}{\partial p_n}
\end{bmatrix},
$$

and the vectors $\boldsymbol{h} = [1 \ \ h_3 \ \ h_4]^\top$ and $\tilde{\boldsymbol{h}} = [h_3' \ \ h_4' \ \ h_3' h_4 - h_3 h_4']^\top$ (recall that $h_2(y) \equiv 1$, so that $h_2'(y) \equiv 0$ and $h_j'(y) h_2(y) \equiv h_j'(y)$, $j = 3, 4$). $\boldsymbol{B}$ is $\frac{1}{2}n(n-1) \times 3$, $\boldsymbol{C}$ is $\frac{1}{2}n(n-1) \times 3$, $\boldsymbol{h}$ is 3×1, and $\tilde{\boldsymbol{h}}$ is 3×1. As before, symmetry can be written in compact matrix notation as $\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{C}\boldsymbol{h}$. Premultiply both sides by $\boldsymbol{B}^\top$ to obtain $\boldsymbol{B}^\top \boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{B}^\top \boldsymbol{C}\boldsymbol{h}$. The 3×3 matrix $\boldsymbol{B}^\top \boldsymbol{B}$ is symmetric, positive definite, so that $\tilde{\boldsymbol{h}} = (\boldsymbol{B}^\top \boldsymbol{B})^{-1} \boldsymbol{B}^\top \boldsymbol{C}\boldsymbol{h} \equiv \boldsymbol{D}\boldsymbol{h}$.

The vectors $\tilde{\boldsymbol{h}}$ and $\boldsymbol{h}$ depend on y but not on $\boldsymbol{p}$, whereas the matrix $\boldsymbol{D}$ can only depend on $\boldsymbol{p}$ and not on y. It follows that all of the elements of $\boldsymbol{D}$ must be constants independent of $\boldsymbol{p}$ and y. That is, the solution to this constrained system of odes can only be a function of y and not $\boldsymbol{p}$.

The implications of symmetry on the income functions can now be written as,

$$\begin{aligned} h_3'(y) &= d_{11} + d_{12}h_3(y) + d_{13}h_4(y), \\ h_4'(y) &= d_{21} + d_{22}h_3(y) + d_{23}h_4(y), \\ h_3(y)h_4'(y) - h_3'(y)h_4(y) &= d_{31} + d_{32}h_3(y) + d_{33}h_4(y), \end{aligned} \tag{46}$$

where the $\{d_{ij}\}$ are constants that cannot all be zero in any given equation (again, by full rank of the demand system). The first two equations form a complete system of linear odes with constant coefficients. This system is constrained by the third equation, which restricts the $\{d_{ij}\}$.

To solve this system of odes, differentiate the first equation and substitute out $h_4'(y)$ and then $h_4(y)$,

$$\begin{aligned} h_3''(y) &= d_{12}h_3'(y) + d_{13}h_4'(y) \\ &= d_{12}h_3'(y) + d_{13}[d_{21} + d_{22}h_3(y) + d_{23}h_4(y)] \\ &= d_{13}d_{21} + d_{12}h_3'(y) + d_{13}d_{22}h_3(y) + d_{23}\left[h_3'(y) - d_{11} - d_{12}h_3(y)\right] \\ &= d_{13}d_{21} - d_{22}d_{11} + (d_{11} + d_{22})h_3'(y) + (d_{13}d_{22} - d_{23}d_{12})h_3(y). \end{aligned}$$

The homogeneous part is,

$$h_3''(y) - (d_{11} + d_{22})h_3'(y) - (d_{13}d_{22} - d_{23}d_{12})h_3(y) = 0,$$

with characteristic equation,

$$\lambda^2 - (d_{11} + d_{22})\lambda - (d_{13}d_{22} - d_{23}d_{12}) = 0,$$

and characteristic roots

$$\lambda = \frac{1}{2}\left[d_{11} + d_{12} \pm \sqrt{(d_{11} + d_{12})^2 + 4(d_{13}d_{22} - d_{23}d_{12})}\right].$$

If $\lambda = 0$ is the only root, then the complete solution is

$$\begin{aligned} h_3(y) &= a_1 + b_1 y + c_1 y^2, \\ h_4(y) &= a_2 + b_2 y + c_2 y^2. \end{aligned}$$

We prove that this is the only possibility.

With distinct non-vanishing roots, the complete solution for the odes is

$$\begin{aligned} h_3(y) &= a_1 + b_1 e^{\lambda_1 y} + c_1 e^{\lambda_2 y}, \\ h_4(y) &= a_2 + b_2 e^{\lambda_1 y} + c_2 e^{\lambda_2 y}. \end{aligned}$$

The second income function, $h_2(y) \equiv 1$, hence, WLOG set $h_3(y) = e^{\lambda_1 y}$ and $h_4(y) = e^{\lambda_2 y}$ by the linear independence of $\{1, e^{\lambda_1 y}, e^{\lambda_2 y}\}$, $\forall\ \lambda_1 \neq \lambda_2 \neq 0$.

The equation for $h_3h_4' - h_3'h_4$ then is

$$(\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)y} = d_{31} + d_{32}e^{\lambda_1 y} + d_{33}e^{\lambda_2 y},$$

where $\lambda_2 - \lambda_1 = \sqrt{(d_{11} + d_{12})^2 + 4(d_{13}d_{22} - d_{23}d_{12})} \neq 0$ and $\lambda_1 + \lambda_2 = d_{11} + d_{12} \neq \lambda_1 \neq \lambda_2$, a contradiction of the linear independence of $\{1, e^{\lambda_1 y}, e^{\lambda_2 y}, e^{(\lambda_1+\lambda_2)y}\}\ \forall\ (\lambda_1, \lambda_2) \neq (0,0)$.

Hence, the characteristic roots must be equal, $\lambda = \frac{1}{2}(d_{11} + d_{12})$. If $\lambda \neq 0$, then the complete solution is

$$h_3(y) = a_1 + b_1 e^{\lambda y} + c_1 y e^{\lambda y},$$
$$h_4(y) = a_2 + b_2 e^{\lambda y} + c_2 y e^{\lambda y}.$$

Set $h_3(y) = e^{\lambda y}$ and $h_4(y) = ye^{\lambda y}$, WLOG, by the linear independence of $\{1, e^{\lambda y}, ye^{\lambda y}\}\ \forall\ \lambda \neq 0$. Then the equation for $h_3h_4' - h_3'h_4$ is

$$e^{2\lambda y} = d_{31} + d_{32}e^{\lambda y} + d_{33}ye^{\lambda y},$$

which is a contradiction of the linear independence of $\{1, e^{\lambda y}, ye^{\lambda y}, e^{2\lambda y}\}$ $\forall\ \lambda \neq 0$.

Hence, only a repeated vanishing root is possible and

$$\frac{\partial y}{\partial \boldsymbol{p}} = \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3 y + \boldsymbol{\alpha}_4 y^2,$$

again, where $y(\boldsymbol{p}, u) = f(\tilde{e}(\boldsymbol{p}, u))$. This has exactly the same form as a nominal income full rank QES. Consequently, the symmetry argument of van Daal and Merkies (1989) applies in tact, which implies

$$\frac{\partial}{\partial \boldsymbol{p}}\left[\frac{f(\tilde{e}(\boldsymbol{p}, u)) - \beta_1(\boldsymbol{p})}{\beta_2(\boldsymbol{p})}\right] = \left[\theta(\beta_3(\boldsymbol{p})) + \left(\frac{f(\tilde{e}(\boldsymbol{p}, u)) - \beta_1(\boldsymbol{p})}{\beta_2(\boldsymbol{p})}\right)^2\right]\frac{\partial \beta_3(\boldsymbol{p})}{\partial \boldsymbol{p}},$$

for some $\beta_1, \beta_2, \beta_3 : \mathbb{R}^n_{++} \to \mathbb{R}$, and some $\theta : \mathbb{R} \to \mathbb{R}$. As before, the change of variables to $z(\boldsymbol{p}, u) = [f(\tilde{e}(\boldsymbol{p}, u)) - \beta_1(\boldsymbol{p})]/\beta_2(\boldsymbol{p})$ reduces this further to

$$\frac{\partial z(\boldsymbol{p}, u)}{\partial \boldsymbol{p}} = \left[\theta(\beta_3(\boldsymbol{p})) + z(\boldsymbol{p}, u)^2\right]\frac{\partial \beta_3(\boldsymbol{p})}{\partial \boldsymbol{p}}.$$

which leads again to the separable function $w(\beta_3(\boldsymbol{p}), u)$ defined in Eq. (26). Hence, the solution for indirect preferences of all full rank Lewbel systems is precisely the same as that obtained for all full rank Gorman systems, but with no restriction on the functional form for $f(\tilde{e})$. ■

Thus, *every* Gorman and Lewbel demand system is a special case of a generalized quadratic expenditure system (GQES). It is useful to emphasize that, in all full rank Lewbel systems, $\{\alpha, \beta, \gamma, \delta, \eta\}$ will depend on no more than three linearly independent, 0° homogeneous price indices, implying that the maximum rank is four.

In general, the rank and structure of a GQES depends on choices for the function f, the price indices $\{\alpha, \beta, \gamma, \delta, \eta, \pi\}$, and the function θ.

Given specific choices for up to three (four) price indices in a Gorman (Lewbel) system, the function f, and when $K = 3$ ($K = 4$) the function θ (or equivalently, the implicit function w), the demand system and associated indirect preferences are completely specified without any need to ever revisit integrability.[12] This complete characterization accommodates the calculation of exact welfare measures, both in the aggregate and for specific consumer groups of interest, as well as many other valuations that are typically of interest in applied research. It is worth emphasizing the fundamental implication of this result: *The "only relevant difference" between a full rank Gorman and a Lewbel system is the choice of functional form for f.*

8. Conclusions

Common reasons for the choice of functional form for demand analysis include parsimony, ease of estimation and interpretation, generality, flexibility, aggregation, and consistency with economic theory. Since the path-breaking papers of Gorman, flexibility and aggregation have guided much of the development and application of applied demand analysis. The rank of Engel curves is a central feature of this research. It is a routine practice to impose the theoretical properties associated with Slutsky symmetry and negativity, homogeneity, and adding up. This chapter shows how to construct *any* GQES demand system, without the need to revisit questions of integrability of the demand equations or the structure and functional form of the implied indirect preference functions.

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[12] In general, $\theta'(x) \neq 0$ "complicates the demand equations while adding nothing to either income or price flexibility, so demands with $[\theta(x) \neq \lambda]$ are not likely to be of much practical interest" (Lewbel, 1987a, p. 1454). This argument is repeated in Lewbel (1990, p. 292). Although one may or may not agree with this claim about the practical implementation of a Gorman or Lewbel system, the set of all models in this class includes any (smooth) $\theta : \mathbb{R} \to \mathbb{R}$, not just the constant functions. This is a substantially larger class of demand systems and preference functions.

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