Duality Theory for Variable Costs in Joint Production

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Abstract

Duality methods for incomplete systems of consumer demand equations are adapted to the dual structure of variable cost functions in joint production. This allows the identification of necessary and sufficient restrictions on technology and cost so that the conditional factor demands can be written as functions of input prices, fixed inputs, and cost. These are observable when the variable inputs are chosen and committed to production, hence the identified restrictions allow *ex ante* conditional demands to be studied using observable data. This class of production technologies is consistent with all von Neumann-Morgenstern utility functions when *ex post* production and/or output prices are uncertain.

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Analysis of multi-product behavior of firms is common in agricultural economics. Techniques of analysis might be based on the distance or production functions, or profit, revenue, or cost functions (Färe and Primont 1995; Just, Zilberman, and Hochman 1983; Shumway 1983, Lopez 1983; Akridge and Hertel 1986). There is a large literature on functional structure and duality that helps guide empirical formulations and testing based on concepts of non-jointness and separability (Lau 1972, 1978; Blackorby, Primont and Russell 1978; Chambers 1984, 1988). For example, separability in some partition of inputs or outputs often results in separability in a similar partition of prices so long as aggregator functions are homothetic (e.g., Blackorby, Primont and Russell 1977; Lau 1978). This allows a researcher to test hypotheses about the structure of technology using cost or profit functions (Shumway 1983). Similarly, the implications of non-jointness often reduce to some form of additivity (Hall 1973; Kohli 1983). Such restrictions on technology guide empiricists as they think about aggregation based on functional structure.

In this article, an issue of functional structure is considered that is somewhat non-standard but useful to empirical work. The question considered is: "When can conventional short-run cost minimizing factor demands a) $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z})$ be written as b) $\boldsymbol{x} = \tilde{\boldsymbol{X}}(\boldsymbol{w}, c, \boldsymbol{z})$, where $\boldsymbol{X}$ and $\tilde{\boldsymbol{X}}$ are vector valued functions, $\boldsymbol{w}$ the corresponding vector of input prices, $\boldsymbol{z}$ is a vector of fixed inputs, $\boldsymbol{y}$ is a vector of outputs, and c is cost?" More precisely, what restrictions on technology, and hence costs, imply that the conditional factor demands can be written as functions of input prices, fixed inputs, and cost rather than the more standard representation in a)?

Interest in answering this question comes from two sources. First, by analogy with Gorman's theory of exact aggregation, if there is cost heterogeneity, it will be natural to think of conditional input demands as dependent on c just as consumer demands depend on income or expenditure.[1]

The second reason is more involved. There is a fairly large literature which proposes solutions to the specification of *ex ante* cost functions when output is uncertain under potentially risk-averse behavior (e.g., Pope and Chavas 1994; Pope and Just 1995; Chambers and Quiggin 2000; Chavas 2008). The essential problem is that if inputs are applied

[1] Heterogeneity across firms in variable costs of production can be extended easily to heterogeneity in fixed inputs along the lines of Lau (1982) – see LaFrance and Pope (2009a) for additional details.

ex ante under stochastic production, then the outputs in a) can't be observed. One approach is to make the assumptions required such that the *ex ante* cost function exists in an empirically convenient form. For example, given random supply shocks ε_i of the form

$$y_i = \bar{y}_i + H_i(\bar{\boldsymbol{y}}, \boldsymbol{z}, \varepsilon_i),\ E\left[H_i(\bar{\boldsymbol{y}}, \boldsymbol{z}, \varepsilon_i) \mid \boldsymbol{x}, \bar{\boldsymbol{y}}, \boldsymbol{z}\right] = 0,\ i = 1, \cdots, n_y, \tag{1}$$

and the existence of a joint production transformation function, $F(\boldsymbol{x}, \bar{\boldsymbol{y}}, \boldsymbol{z}) \leq 0$, defined over variable inputs, $\boldsymbol{x}$, planned outputs, $\bar{\boldsymbol{y}}$, and fixed inputs, $\boldsymbol{z}$, then the reasoning in Pope and Chavas (1994) implies the existence of a cost function in which $\bar{\boldsymbol{y}}$ replaces $\boldsymbol{y}$. That is, minimizing the variable cost of planned outputs yields

$$c = C(\boldsymbol{w}, \bar{\boldsymbol{y}}, \boldsymbol{z}) \equiv \min_{\boldsymbol{x} \geq \boldsymbol{0}} \left\{ \boldsymbol{w}^{\top} \boldsymbol{x} : F(\boldsymbol{x}, \bar{\boldsymbol{y}}, \boldsymbol{z}) \leq 0 \right\}, \tag{2}$$

where the symbol $^{\top}$ denotes vector and matrix transposition. It can be shown that cost minimization in terms of planned outputs holds for <u>*all*</u> von Neumann-Morgenstern utility functions in both static and dynamic environments. The conditional factor demands, $\boldsymbol{X}(\boldsymbol{w}, \bar{\boldsymbol{y}}, \boldsymbol{z})$, will continue to depend on the unobservable variables, $\bar{\boldsymbol{y}}$. However, these input demand functions only depend on $(\boldsymbol{w}, \boldsymbol{z}, c)$, all of which are observable, when a) reduces to b). Thus, the restrictions we seek are those that allow *ex ante* conditional demands to be studied using only observable variables. Dropping the $\bar{\boldsymbol{y}}$ notation by considering $\boldsymbol{y}$ to be planned output, the question is when can ordinary cost minimizing factor demands $\boldsymbol{X}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z})$ be written as $\tilde{\boldsymbol{X}}(\boldsymbol{w}, c, \boldsymbol{z})$?

In this article, we find that the necessary and sufficient condition for a) to reduce to b), $\boldsymbol{X}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}) = \tilde{\boldsymbol{X}}(\boldsymbol{w}, C(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}), \boldsymbol{z})$, is that $c = C(\boldsymbol{w}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) \Leftrightarrow F(\boldsymbol{x}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z}))$. That is, outputs must be weakly separable from variable inputs in the joint production technology, or equivalently, outputs must be weakly separable from variable input prices in the cost function.[2] The argument, complete with technical details, is presented in the companion Appendix to this paper, available on AgEconSearch as LaFrance and Pope (2009b). As illustrated in the penultimate section, when this separability condition holds, empirical work is simplified substantially and also may be much more robust.

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[2] This implies that marginal rates of product transformation are independent of variable inputs. However, as is shown in the section on empirical implementation, the variable inputs are <u>*not*</u> weakly separable from outputs in the joint production transformation function, and the variable input prices are <u>*not*</u> weakly separable from outputs in the variable cost function.

Duality and the Main Result

The neoclassical model of conditional demands for variable inputs with joint production, fixed inputs, and production uncertainty is

$$\boldsymbol{X}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z}) = \arg\min\left\{\boldsymbol{w}^{\top}\boldsymbol{x} : F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \le 0,\ \boldsymbol{x} \ge \boldsymbol{0}\right\}, \tag{3}$$

where $\boldsymbol{x}$ is an n_x–vector of positive variable inputs with corresponding positive prices, $\boldsymbol{w}$, $\boldsymbol{y}$ is an n_y–vector of planned outputs, $\boldsymbol{z}$ is an n_z–vector of fixed inputs, F is the real valued transformation function that defines the boundary of a closed, convex production possibilities set with free disposal in the inputs and the outputs, $\boldsymbol{X}$ maps variable input prices, planned outputs, and fixed inputs into variable input demand functions, and $C(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z}) \equiv \boldsymbol{w}^{\top}\boldsymbol{X}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z})$, is the positive-valued variable cost function.[3] By Shephard's Lemma, we have

$$\boldsymbol{X}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z}) = \nabla_{\boldsymbol{w}} C(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z}) \equiv (\partial C/\partial w_1, \cdots, \partial C/\partial w_{n_x})^{\top}. \tag{4}$$

Recall that $\boldsymbol{X}$ is homogeneous of degree zero in $\boldsymbol{w}$ by the derivative property of homogeneous functions. Integrating with respect to $\boldsymbol{w}$ to recover the variable cost function, we obtain

$$c = C(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z}) \equiv \tilde{C}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})), \tag{5}$$

where $\theta : \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Z}} \to \mathbb{R}$ is the *constant of integration*.[4] In the present case, this means that θ is constant with respect to $\boldsymbol{w}$. In general, however, θ is a function of both $\boldsymbol{y}$ and $\boldsymbol{z}$ and its structure cannot be identified from the variable input demands because it captures that

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[3] The paper focuses on interior solutions and smooth functions. The results can be extended in the standard way to corner solutions by a continuous extension of F or C to the boundary of the strictly positive orthant in $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ or $(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z})$ space (see, e.g., Blackorby, Primont, and Russell 1978). Smoothness can be relaxed to twice continuous differentiability with no change in the arguments that follow.

[4] A reviewer asked, "Why is the constant of integration not shown as additive?" To answer this question, consider the following example. Given the differential equation, $y'(x) = a(x) + by(x)$, apply the integrating factor e^{-bx} to rewrite it in the form $\frac{d}{dx}[y(x)e^{-bx}] = [y'(x) - by(x)]e^{-bx} = a(x)e^{-bx}$. Integrating and solving for $y(x)$ yields $y(x) = [\alpha(x) + c]e^{bx}$, where $\alpha(x) = \int a(x)e^{-bx}dx$. Similar reasoning applied to many cases shows that, in general, once one recovers the cost function through integration with respect to $\boldsymbol{w}$, the *constant of integration*, $\theta(\boldsymbol{y},\boldsymbol{z})$, will not be additively separable from $\boldsymbol{w}$.

part of the joint production process relating to the fixed inputs and the outputs that is separable from the variable inputs.[5] Though the constant of integration is often written additively, in some cases (e.g., the Cobb-Douglas) it enters costs multiplicatively, and in others it enters nonlinearly. In particular, the cost function is linearly homogeneous in input prices, while θ is independent of prices. Consequently, it is impossible for $\tilde{C}$ to be additively separable in θ. The example in the empirical implementation section clearly illustrates this issue.

Under standard and well-known conditions, the variable cost function is strictly decreasing in $\boldsymbol{z}$, strictly increasing in $\boldsymbol{y}$, jointly convex in $(\boldsymbol{y}, \boldsymbol{z})$, and increasing, concave and homogeneous of degree one in $\boldsymbol{w}$. We are free to choose the *sign* of θ so that, with no loss of generality, $\partial\tilde{C}/\partial\theta > 0$.[6]

Because $\tilde{C}$ is strictly increasing in θ, it has a unique inverse, $\theta = \gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, c)$, where $\gamma : \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Z}} \times \mathbb{R}_+ \to \mathbb{R}$ is the real-valued inverse of $\tilde{C}$ with respect to θ. The function $\gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, c)$ is a *quasi-indirect production transformation function,* analogous to the quasi-indirect utility function of consumer theory (Hausman 1981; Epstein 1982; LaFrance 1985, 1986, 1990, 2004; LaFrance and Hanemann 1989; von Haefen 2002). Because γ is the inverse of $\tilde{C}$ with respect to θ, it only *partially* reflects technology; hence the qualifier "quasi" is used. That is, in (5) above, the variable portion of technology is embedded in the properties of $\tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \theta)$, while the properties of $\theta(\boldsymbol{y}, \boldsymbol{z})$ cannot be identified from those of $\tilde{C}$ or $\boldsymbol{X}$. For all interior and feasible $(\boldsymbol{y}, \boldsymbol{z})$, the function γ is strictly increasing in c, strictly decreasing and quasi-convex in $\boldsymbol{w}$, and positively homogeneous of degree zero in ($\boldsymbol{w}$, c).

At the heart of many explanations and applications of duality theory in economics are identities involving inverses. Well-known examples involve the expenditure function, indirect utility function, cost function, indirect profit function, and indirect production function. These two identities are simple implications of the inverse function theorem:

[5] We elucidate this point in detail below.

[6] To see this, define $\tilde{\theta} = -\theta$, so that $\hat{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \tilde{\theta}) \equiv \tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, -\tilde{\theta}) \equiv \tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \theta)$. The composite function theorem gives $\partial\hat{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \tilde{\theta})/\partial\tilde{\theta} \equiv -\partial\tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, -\tilde{\theta})/\partial\theta \equiv -\partial\tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \theta)/\partial\theta$. Thus, reversing the sign of θ, which we are free to do whenever convenient (precisely because it is an arbitrary function that cannot be identified or recovered from the variable input demand equations), also reverses the sign of the derivative of the variable cost function with respect to θ.

$$c \equiv \tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, c)); \tag{6}$$

and

$$\theta \equiv \gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \theta)). \tag{7}$$

This lets one write the conditional demands for the variable inputs as

$$\boldsymbol{x} = \nabla_{\boldsymbol{w}} \tilde{C} \equiv \boldsymbol{G}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, c). \tag{8}$$

Thus, (8) gives the rationale for writing the factor demands as a function of c, as well as $(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z})$. Thus, given the above regularity conditions for *F* and *C*, one can _always_ write the system of factor demands as functions of cost.

It is useful to note some duality properties that are derived from γ. Define the *quasi-production transformation function* by

$$\upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \equiv \min_{\boldsymbol{w} \geq \boldsymbol{0}} \left\{ \gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}^{\top} \boldsymbol{x}) \right\}. \tag{9}$$

As before with γ, the terminology *quasi-production transformation function* indicates that $\upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ only reveals _part_ of the structure of the joint production process. In particular, it cannot, and does not, reveal anything about $\theta(\boldsymbol{y}, \boldsymbol{z})$. Also as before, this is analogous to the situation where one only recovers part of a direct utility function when analyzing the market demands for a subset of consumption goods. If there is only one output and no fixed inputs, then $c = C(\boldsymbol{w}, y)$, $\theta \equiv y = \gamma(\boldsymbol{w}, c)$, and the complete technology is obtained from the minimization problem, $y = F(\boldsymbol{x}) = \min_{\boldsymbol{w} \geq \boldsymbol{0}} \gamma(\boldsymbol{w}, \boldsymbol{w}^{\top} \boldsymbol{x})$.[7]

For the general case, the identity $\theta(\boldsymbol{y}, \boldsymbol{z}) \equiv \gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})))$, which is (7), implies

$$\theta(\boldsymbol{y}, \boldsymbol{z}) \equiv \gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z}))) \geq \min_{\boldsymbol{w} \geq \boldsymbol{0}} \left\{ \gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}^{\top} \boldsymbol{x}) \right\} \equiv \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}), \tag{10}$$

for all interior and feasible $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$. The inequality in (10) follows from the fact that $\theta(\boldsymbol{y}, \boldsymbol{z})$ is feasible but is not necessarily optimal in the minimization problem. The part of $F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ that is not contained in $\upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is given by (see Diewert (1975), Epstein (1975), Hausman (1981), and LaFrance and Hanemman (1989) in the case of consumer

[7] For _any_ continuous function, $H(\boldsymbol{x}, y)$, $H : \mathcal{X} \times \mathcal{Y} \subseteq \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, the scalar variable y is _always_ weakly separable from the variables $\boldsymbol{x}$ with the identity as aggregator (Blackorby, Primont and Russell, 1978).

theory),[8]

$$F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \equiv \tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})). \tag{11}$$

That is, the quasi-production transformation function is the solution, $\theta = \upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$, of the implicit function, $\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\theta) = 0,$ in other words, $\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})) \equiv 0.$ It is shown in the Appendix that $\upsilon(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ in (9) conveys full information about the marginal rates of substitution between variable inputs but only partially so for outputs and fixed inputs. This is again analogous to the situation in consumption theory when one analyzes only a subset of the goods purchased and consumed.

It is therefore clear that separability of the technology is the key property required to establish when the variable input demands can be written as $\tilde{\boldsymbol{X}}(\boldsymbol{w},c,\boldsymbol{z})$. If the variable cost function is of the form $c = \tilde{C}(\boldsymbol{w},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z}))$, then γ depends on $\boldsymbol{y}$ only indirectly and $\theta = \gamma(\boldsymbol{w},\boldsymbol{z},c)$. This implies that $\min_{\boldsymbol{w}\geq\boldsymbol{0}} \gamma(\boldsymbol{w},\boldsymbol{z},\boldsymbol{w}^{\top}\boldsymbol{x}) = \upsilon(\boldsymbol{x},\boldsymbol{z})$, and hence $\upsilon(\boldsymbol{x},\boldsymbol{z}) \leq \theta(\boldsymbol{y},\boldsymbol{z})$ for all feasible $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$. Clearly, $\boldsymbol{x} = \tilde{\boldsymbol{X}}(\boldsymbol{w},c,\boldsymbol{z}),$ either by minimizing $\boldsymbol{w}^{\top}\boldsymbol{x}$ subject to $\theta - \upsilon(\boldsymbol{x},\boldsymbol{z}) \leq 0$ and substituting $\theta = \gamma(\boldsymbol{w},\boldsymbol{z},c)$ in the resulting variable input demands, $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{w},\boldsymbol{z},\theta),$ or by applying Shephard's Lemma to $\tilde{C}(\boldsymbol{w},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z}))$ and making the same substitution. This leads to the following result.

Proposition*: The following functional structures are equivalent:*

$$\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z}) \equiv \tilde{\boldsymbol{X}}(\boldsymbol{w},c,\boldsymbol{z}); \tag{12}$$

$$c = C(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z}) \equiv \tilde{C}(\boldsymbol{w},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})); \tag{13}$$

and

$$0 = F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \equiv \tilde{F}(\boldsymbol{x},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})). \tag{14}$$

Proof: See the Appendix.

Therefore, separability of outputs from variable inputs in technology – equivalently, outputs separable from variable input prices in the variable cost function – is necessary and sufficient for the variable input demands to be representable in terms of cost rather than outputs. This property is commonly imposed in studies that aggregate across outputs to form a single aggregate output – e.g., the many studies of aggregate U.S. agricultural

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[8] Whether or not $\theta : \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ *exists* is not an issue in this context. One could *always* define the function $\tilde{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\theta(\boldsymbol{y},\boldsymbol{z})) \equiv F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) + \theta(\boldsymbol{y},\boldsymbol{z}),$ with $\theta(\boldsymbol{y},\boldsymbol{z}) \equiv 0$ for all feasible $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ satisfying $F(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = 0.$ The essential question is, "Does $\boldsymbol{y}$ <u>only</u> enter $\tilde{F}$ through the separable aggregator θ?"

output. Once this assumption has been made, the conditional input demands can be represented conveniently in terms of cost without the need to use output explicitly. This avoids issues such as whether outputs are random, and if so, how best to model the formation of producer expectations. The next section illustrates this result for empirical applications.

Empirical Implementation

The simplest cost function that illustrates the main result has the multiplicative form

$$c = \tilde{C}(\boldsymbol{w}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) = \varsigma(\boldsymbol{w}, \boldsymbol{z})\theta(\boldsymbol{y}, \boldsymbol{z}), \tag{15}$$

where ς is positive-valued, concave and homogeneous of degree one in $\boldsymbol{w}$. This functional form imposes the additional (very strong) restriction that the variable input prices are separable from the outputs in *C*. However, its simplicity is both intuitively appealing and instructive. By Shephard's Lemma,

$$\boldsymbol{x} = \frac{\partial \varsigma(\boldsymbol{w}, \boldsymbol{z})}{\partial \boldsymbol{w}} \theta(\boldsymbol{y}, \boldsymbol{z}) = \frac{\partial \varsigma(\boldsymbol{w}, \boldsymbol{z})}{\partial \boldsymbol{w}} \frac{c}{\varsigma(\boldsymbol{w}, \boldsymbol{z})} = \tilde{\boldsymbol{X}}(\boldsymbol{w}, c, \boldsymbol{z}), \tag{16}$$

while $\theta = \gamma(\boldsymbol{w}, \boldsymbol{z}, c) = c/\varsigma(\boldsymbol{w}, \boldsymbol{z})$ and $\upsilon(\boldsymbol{x}, \boldsymbol{z}) = \min_{\boldsymbol{w} \geq \boldsymbol{0}} \left\{ \boldsymbol{w}^{\top}\boldsymbol{x} / \varsigma(\boldsymbol{w}, \boldsymbol{z}) \right\}$. The technology is found by setting $\theta(\boldsymbol{y}, \boldsymbol{z}) - \upsilon(\boldsymbol{x}, \boldsymbol{z}) = 0$ under technical efficiency, or for an arbitrary continuous and monotonic $\tilde{F} : \mathbb{R} \to \mathbb{R}$, $\tilde{F}(\theta(\boldsymbol{y}, \boldsymbol{z}) - \upsilon(\boldsymbol{x}, \boldsymbol{z})) = 0$.

Though this example is instructive, more general models are useful in empirical work. The remainder of this section focuses on a class of variable input demand models that is flexible and consistent with the proposition. Each member of this class aggregates exactly across heterogeneous costs and fixed inputs and can achieve maximum rank – either full rank three or full rank four, (Lewbel 1989, 1990; LaFrance and Pope 2009a) – of any variable input demand system that satisfies these properties. This class of models is quite complex relative to standard applications in production economics – e.g., the translog, normalized quadratic, or generalized Leontief. However, the models in this class allow empirical researchers to estimate variable factor demands in *ex ante* form and to nest both the rank and the functional form of the demand equations.

Let the real-valued function G satisfy $G' > 0$, $G'' \geq 0$, let the strictly positive-valued function $\pi(\boldsymbol{w}, \boldsymbol{z})$ be increasing, concave, and homogeneous of degree one in $\boldsymbol{w}$, let

α, β, δ be nonnegative real-valued functions that are homogeneous of degree zero in $\boldsymbol{w}$, and let $\theta(\boldsymbol{y}, \boldsymbol{z})$ satisfy $\nabla_{\boldsymbol{y}} \theta(\boldsymbol{y}, \boldsymbol{z}) \gg \mathbf{0}$ and $\nabla^2_{\boldsymbol{y}\boldsymbol{y}^\top} \theta(\boldsymbol{y}, \boldsymbol{z})$ is symmetric and positive semi-definite.[9] Define the variable cost function by

$$C(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}) = \pi(\boldsymbol{w}, \boldsymbol{z}) \times G\left(\frac{\beta(\boldsymbol{w}, \boldsymbol{z})}{\left[\alpha(\boldsymbol{w}, \boldsymbol{z}) - \theta(\boldsymbol{y}, \boldsymbol{z})\right]} - \delta(\boldsymbol{w}, \boldsymbol{z}) \right). \tag{17}$$

Equivalently, define the quasi-indirect production transformation function by[10]

$$\theta = \tilde{\gamma}(\boldsymbol{w}, \boldsymbol{z}, c) = \alpha(\boldsymbol{w}, \boldsymbol{z}) - \frac{\beta(\boldsymbol{w}, \boldsymbol{z})}{\left[G^{-1}\left(c/\pi(\boldsymbol{w}, \boldsymbol{z})\right) + \delta(\boldsymbol{w}, \boldsymbol{z})\right]}. \tag{18}$$

Since there are four functions of $\boldsymbol{w}$ in the variable cost function, variable input prices are not separable from outputs in *C*. On the other hand, there is only one aggregator for $\boldsymbol{y}$, so that outputs are separable from variable input prices in *C*.

Applying Shephard's lemma to (17) gives the variable input demands as

$$\begin{aligned} \boldsymbol{X}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}) &= G\left(\frac{\beta(\boldsymbol{w}, \boldsymbol{z})}{\left[\alpha(\boldsymbol{w}, \boldsymbol{z}) - \theta(\boldsymbol{y}, \boldsymbol{z})\right]} - \delta(\boldsymbol{w}, \boldsymbol{z}) \right) \nabla_{\boldsymbol{w}} \pi(\boldsymbol{w}, \boldsymbol{z}) \\ &+ \pi(\boldsymbol{w}, \boldsymbol{z}) G'\left(\frac{\beta(\boldsymbol{w}, \boldsymbol{z})}{\left[\alpha(\boldsymbol{w}, \boldsymbol{z}) - \theta(\boldsymbol{y}, \boldsymbol{z})\right]} - \delta(\boldsymbol{w}, \boldsymbol{z}) \right) \left\{ \frac{\nabla_{\boldsymbol{w}} \beta(\boldsymbol{w}, \boldsymbol{z})}{\left[\alpha(\boldsymbol{w}, \boldsymbol{z}) - \theta(\boldsymbol{y}, \boldsymbol{z})\right]} \right. \\ &\left. + \frac{\beta(\boldsymbol{w}, \boldsymbol{z}) \nabla_{\boldsymbol{w}} \alpha(\boldsymbol{w}, \boldsymbol{z})}{\left[\alpha(\boldsymbol{w}, \boldsymbol{z}) - \theta(\boldsymbol{y}, \boldsymbol{z})\right]^2} - \nabla_{\boldsymbol{w}} \delta(\boldsymbol{w}, \boldsymbol{z}) \right\}. \end{aligned} \tag{19}$$

Substituting the right-hand-side of (18) into (19) and rearranging terms then yields the variable input demand equations in the alternative form given in the proposition,

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[9] The restrictions on *G* and θ with respect to $\boldsymbol{y}$ together with the monotonicity condition $\alpha(\boldsymbol{w}, \boldsymbol{z}) > \theta(\boldsymbol{y}, \boldsymbol{z})$ are sufficient for the variable cost function to be increasing and convex in outputs. If $\boldsymbol{z}$ is a vector of fixed inputs, then the conditions for *C* to be decreasing and convex in $\boldsymbol{z}$ are complex and omitted at this juncture.

[10] Except for special cases, this class of variable cost functions does not have closed form solutions for the joint production transformation function.

$$
\begin{aligned}
\tilde{\boldsymbol{X}}(\boldsymbol{w},c,\boldsymbol{z}) &= \left(\frac{c}{\pi(\boldsymbol{w},\boldsymbol{z})}\right)\nabla_{\boldsymbol{w}}\pi(\boldsymbol{w},\boldsymbol{z}) \\
&+ G'\left(G^{-1}\left(c/\pi(\boldsymbol{w},\boldsymbol{z})\right)\right)\pi(\boldsymbol{w},\boldsymbol{z})\left\{\left(\frac{G^{-1}\left(c/\pi(\boldsymbol{w},\boldsymbol{z})\right)+\delta(\boldsymbol{w},\boldsymbol{z})}{\beta(\boldsymbol{w},\boldsymbol{z})}\right)\nabla_{\boldsymbol{w}}\beta(\boldsymbol{w},\boldsymbol{z}) \right. \\
&\left. + \frac{\left[G^{-1}\left(c/\pi(\boldsymbol{w},\boldsymbol{z})\right)+\delta(\boldsymbol{w},\boldsymbol{z})\right]^2}{\beta(\boldsymbol{w},\boldsymbol{z})}\nabla_{\boldsymbol{w}}\alpha(\boldsymbol{w},\boldsymbol{z}) - \nabla_{\boldsymbol{w}}\delta(\boldsymbol{w},\boldsymbol{z})\right\}.
\end{aligned}
\tag{20}
$$

Note that there are up to four independent vectors of input price functions, and up to four independent functions of cost on the right-hand-side of (20). This implies that this model encompasses the entire class of exactly aggregable Gorman/Lewbel demand systems, as adapted here to production applications.

Given the general functional forms above, one useful choice of functional form for G^{-1} is a translated Box-Cox transformation, $G^{-1}(x) = (x^{\kappa} - 1 + \kappa)/\kappa$, $\kappa \in \mathbb{R}_+$. This reduces the maximum rank to three,[11] while nesting all members of the Price Independent Generalized Linear and Price Independent Generalized Logarithmic functional forms derived in Muellbauer (1975, 1976). Ball, et al. (2009) contains a detailed discussion and empirical application of this modeling framework to state-level demands for variable inputs in U.S. agriculture. In that research study, the farm wage rate, w_n, is the numeraire, $\pi(\boldsymbol{w},\boldsymbol{z}) = w_n$, and the remaining choices for the price indices are

$$
\begin{aligned}
\alpha(\boldsymbol{w},\boldsymbol{z}) &= \alpha_{00} + \boldsymbol{\alpha}_0^{\mathsf{T}}\boldsymbol{z} + \frac{\boldsymbol{\alpha}_1^{\mathsf{T}}[(\boldsymbol{w}/w_n)^{\lambda} - \boldsymbol{\iota} + \lambda\boldsymbol{\iota}]}{\lambda}, \\
\beta(\boldsymbol{w},\boldsymbol{z}) &= \sqrt{\frac{[(\boldsymbol{w}/w_n)^{\lambda} - \boldsymbol{\iota} + \lambda\boldsymbol{\iota}]^{\mathsf{T}}\boldsymbol{B}[(\boldsymbol{w}/w_n)^{\lambda} - \boldsymbol{\iota} + \lambda\boldsymbol{\iota}]}{\lambda^2} + 2\frac{\boldsymbol{\beta}_1^{\mathsf{T}}[(\boldsymbol{w}/w_n)^{\lambda} - \boldsymbol{\iota} + \lambda\boldsymbol{\iota}]}{\lambda} + 1}, \\
\delta(\boldsymbol{w},\boldsymbol{z}) &= \frac{\delta_{00} + \boldsymbol{\delta}_0^{\mathsf{T}}\boldsymbol{z} + \dfrac{\boldsymbol{\delta}_1^{\mathsf{T}}[(\boldsymbol{w}/w_n)^{\lambda} - \boldsymbol{\iota} + \lambda\boldsymbol{\iota}]}{\lambda}}{\beta(\boldsymbol{w},\boldsymbol{z})},
\end{aligned}
\tag{21}
$$

[11] This restriction on the rank of aggregable demand systems was originally derived by Gorman (1981) and is analyzed in more detail by Lewbel (1989, 1991).

where $\boldsymbol{\iota} = (1, \cdots, 1)^{\intercal}$ is an (n_x–1)–vector of ones, $\lambda \in \mathbb{R}_+$ is a parameter, $\alpha_{00}, \delta_{00} \in \mathbb{R}$ are parameters $\boldsymbol{\alpha}_0, \boldsymbol{\delta}_0$ are n_z–vectors of parameters, $\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\delta}_1$ are (n_x–1)–vectors of parameters, and $\boldsymbol{B}$ a symmetric, positive definite (n_x–1)×(n_x–1) matrix of parameters. These choices conveniently allow one to nest a large class of functional forms for prices that includes, *inter alia*, the logarithmic form of the translog (Christensen, Jorgenson and Lau 1973, 1975), the square root form of the generalized Leontief (Diewert 1971) and Minflex Laurent (Barnett and Lee 1985, and Barnett, Lee and Wolfe 1985), and the linear form of the quadratic and normalized quadratic models (Lau 1976).[12]

Conclusions

An empirically important question concerns when cost-minimizing input demands can be stated in terms of empirically observable *ex ante* data: costs, input prices, and fixed or quasi-fixed inputs. We find that separability of the expected outputs from the variable inputs must occur in technology and equivalently that separability of the expected outputs from the variable input prices must occur in the cost function. If these restrictions are deemed to be too strong, then alternative approaches to cost function formulation must be pursued.

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[12] Ongoing work by the authors involves econometrically estimating this model of variable input demands for thirteen variable inputs (labor, pesticides, fertilizer, fuel and natural gas, electricity, purchased feed, purchased seed, purchased livestock, machinery repairs, building repairs, custom machinery hired, veterinary services, and other materials) in U.S. agriculture at both the state- and national-level. This econometric project includes imposing the restrictions: $\alpha \geq 0$; $\beta > 0$; $G^{-1}(c/\pi) + \delta < 0$; $\kappa \geq 1$; and $\lambda \leq 1$, which can be shown to be necessary and sufficient for essentially global monotonicity and concavity of the variable cost function – i.e., over an open, convex set in the $(\boldsymbol{w}, c, \boldsymbol{z})$ space containing the convex hull of the data.

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Appendix

Let $\boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}_{++}^{n_x}$ be an n_x–vector of variable inputs, let $\boldsymbol{w} \in \mathcal{W} \subseteq \mathbb{R}_{++}^{n_x}$ be an n_x–vector of variable input prices, let $\boldsymbol{y} \in \mathcal{Y} \subseteq \mathbb{R}_{++}^{n_y}$ be an n_y–vector of outputs, let $\boldsymbol{z} \in \mathcal{Z} \subseteq \mathbb{R}_{++}^{n_z}$ be an n_z–vector of fixed inputs, let $F : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ be a transformation function that defines the boundary of a closed, convex production possibilities set with free disposal in inputs and outputs, let $\boldsymbol{X} : \mathcal{W} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$, be an n_x–vector of variable input demand functions, and let $C : \mathcal{W} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}_{++}$ be a variable cost function,

$$c = C(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}) \equiv \min_{\boldsymbol{x}} \left\{ \boldsymbol{w}^{\top} \boldsymbol{x} : F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \le 0, \boldsymbol{x} \ge \boldsymbol{0} \right\} \equiv \boldsymbol{w}^{\top} \boldsymbol{X}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}), \tag{A.1}$$

where the symbol $^{\top}$ denotes vector and matrix transposition. The purpose of this appendix is to prove that short-run cost-minimizing variable input demands, $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z})$, can be written in the form $\boldsymbol{x} = \tilde{\boldsymbol{X}}(\boldsymbol{w}, c, \boldsymbol{z})$ if and only if $c = C(\boldsymbol{w}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) \Leftrightarrow F(\boldsymbol{x}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z}))$. For clarity, we first summarize the basic background material from the man paper.

The main paper explains that the complete solution to the system of partial differential equations, $\boldsymbol{X}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}) = \nabla_{\boldsymbol{w}} C(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z})$, <u>*always*</u> can be written in the general form, $C(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z})) = \tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z}))$, where $\theta : \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ is a function of outputs and fixed inputs but not variable input prices. The function $\tilde{C} : \mathcal{W} \times \mathcal{Y} \times \mathcal{Z} \times \mathbb{R} \to \mathbb{R}$ is strictly increasing in θ (with no loss of generality), so that $\theta = \gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, c)$ – the *quasi-indirect production transformation function* – exists, is continuous, and is strictly increasing in c, where $\gamma : \mathcal{W} \times \mathcal{Y} \times \mathcal{Z} \times \mathbb{R}_{++} \to \mathbb{R}$ is the inverse of $\tilde{C}$ with respect to θ, The inverse function theorem implies

$$c \equiv \tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, c)) \tag{A.2}$$

and

$$\theta \equiv \gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \tilde{C}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \theta)). \tag{A.3}$$

The *quasi-production transformation function* also is defined in the main paper by

$$\upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \equiv \min_{\boldsymbol{w} \ge \boldsymbol{0}} \left\{ \gamma(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}^{\top} \boldsymbol{x}) \right\}. \tag{A.4}$$

The function $\upsilon : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ is the unique solution, $\theta = \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$, to the implicit function, $F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \equiv \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) = 0$, i.e., $\tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})) \equiv 0$, where the function $\theta : \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ reflects the part of F for which the outputs, $\boldsymbol{y}$, and the fixed inputs, $\boldsymbol{z}$, are (weakly) separable from the variable inputs, $\boldsymbol{x}$. The implicit function theorem applied to $\tilde{F}$ gives

$$\nabla_{\boldsymbol{x}} \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = -\frac{\nabla_{\boldsymbol{x}} \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}))}{\nabla_{\theta} \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}))},$$

$$\nabla_{\boldsymbol{y}} \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = -\frac{\nabla_{\boldsymbol{y}} \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}))}{\nabla_{\theta} \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}))}, \tag{A.5}$$

$$\nabla_{\boldsymbol{z}} \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = -\frac{\nabla_{\boldsymbol{z}} \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}))}{\nabla_{\theta} \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}))}.$$

This demonstrates that υ conveys full information on marginal rates of substitution between variable inputs,

$$\frac{\partial \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) / \partial x_i}{\partial \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) / \partial x_j} = \frac{\partial \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})) / \partial x_i}{\partial \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})) / \partial x_j} = \frac{\partial F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) / \partial x_i}{\partial F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) / \partial x_j}, \ \forall \ i, j = 1, \cdots, n_x, \tag{A.6}$$

but only partial information on marginal rates of product transformation between outputs,

$$\begin{aligned} \frac{\partial F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) / \partial y_i}{\partial F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) / \partial y_j} &= \frac{\partial \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) / \partial y_i + \partial \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) / \partial \theta \cdot \partial \theta(\boldsymbol{y}, \boldsymbol{z}) / \partial y_i}{\partial \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) / \partial y_j + \partial \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) / \partial \theta \cdot \partial \theta(\boldsymbol{y}, \boldsymbol{z}) / \partial y_j} \\ &\neq \frac{\partial \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) / \partial y_i}{\partial \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) / \partial y_j}, \ \forall \ i, j = 1, \cdots, n_y, \end{aligned} \tag{A.7}$$

and marginal rates of substitution between fixed inputs,

$$\begin{aligned} \frac{\partial F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) / \partial z_i}{\partial F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) / \partial z_j} &= \frac{\partial \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) / \partial z_i + \partial \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) / \partial \theta \cdot \partial \theta(\boldsymbol{y}, \boldsymbol{z}) / \partial z_i}{\partial \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) / \partial z_j + \partial \tilde{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) / \partial \theta \cdot \partial \theta(\boldsymbol{y}, \boldsymbol{z}) / \partial z_j} \\ &\neq \frac{\partial \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) / \partial z_i}{\partial \upsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) / \partial z_j}, \ \forall \ i, j = 1, \cdots, n_z. \end{aligned} \tag{A.8}$$

This background leads to the following result in the dual structure of variable costs in joint production problems.

***Proposition**: The following functional structures are equivalent:*

$$\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}) \equiv \tilde{\boldsymbol{X}}(\boldsymbol{w}, c, \boldsymbol{z}); \tag{A.9}$$

$$c = C(\boldsymbol{w}, \boldsymbol{y}, \boldsymbol{z}) \equiv \tilde{C}(\boldsymbol{w}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})); \tag{A.10}$$

and

$$0 = F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \equiv \tilde{F}(\boldsymbol{x}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})). \tag{A.11}$$

Proof: (A.10) $\Rightarrow$ (A.9). Differentiating (A.10) with respect to $\boldsymbol{w}$, Shephard's Lemma implies,

$$\boldsymbol{x} = \nabla_{\boldsymbol{w}} \tilde{C}. \tag{A.12}$$

$\tilde{C}$ is strictly monotonic in and has a unique inverse with respect to θ, say $\theta = \tilde{\gamma}(\boldsymbol{w}, \boldsymbol{z}, c)$. Substituting this into (A.12) obtains

$$\boldsymbol{x} = \nabla_{\boldsymbol{w}} \tilde{C}(\boldsymbol{w}, \boldsymbol{z}, \tilde{\gamma}(\boldsymbol{w}, \boldsymbol{z}, c)) \equiv \tilde{\boldsymbol{X}}(\boldsymbol{w}, c, \boldsymbol{z}). \tag{A.13}$$

(A.11) ⇒ (A.9) ⇒ (A.10). If the representation of technology has the separable structure in (A.11), then

$$\arg\min\left\{\boldsymbol{w}^{\top}\boldsymbol{x} : \tilde{F}(\boldsymbol{x}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) \le 0,\ \boldsymbol{x} \ge \boldsymbol{0}\right\} \equiv \tilde{\boldsymbol{X}}(\boldsymbol{w}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})). \tag{A.14}$$

This implies that the variable cost function has the separable structure

$$\boldsymbol{w}^{\top}\tilde{\boldsymbol{X}}(\boldsymbol{w}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})) \equiv \tilde{C}(\boldsymbol{w}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z})). \tag{A.15}$$

(A.10) ⇒ (A.11). Given (A.10), the quasi-production transformation function satisfies

$$\tilde{\upsilon}(\boldsymbol{x}, \boldsymbol{z}) \equiv \min_{\boldsymbol{w} \ge \boldsymbol{0}}\left\{\tilde{\gamma}(\boldsymbol{w}, \boldsymbol{z}, \boldsymbol{w}^{\top}\boldsymbol{x})\right\}. \tag{A.16}$$

The same logic that leads to (10) in the main paper now implies,

$$\theta(\boldsymbol{y}, \boldsymbol{z}) \equiv \gamma(\boldsymbol{x}, \boldsymbol{z}, \tilde{C}(\boldsymbol{x}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z}))) \ge \tilde{\upsilon}(\boldsymbol{x}, \boldsymbol{z}), \tag{A.17}$$

for all interior, feasible $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$, with the boundary of the closed and convex production possibilities set defined by equality on the far right. Since $\tilde{\upsilon}$ is independent of $\boldsymbol{y}$, equations (A.5) and (A.7) imply

$$\frac{\partial F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})/\partial y_i}{\partial F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})/\partial y_j} = \frac{\partial \theta(\boldsymbol{y}, \boldsymbol{z})/\partial y_i}{\partial \theta(\boldsymbol{y}, \boldsymbol{z})/\partial y_j},\ \forall\ i, j = 1, \cdots, n_y. \tag{A.18}$$

Hence, the marginal rates of transformation between outputs are independent of variable inputs,

$$\frac{\partial}{\partial x_k}\left(\frac{\partial F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})/\partial y_i}{\partial F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})/\partial y_j}\right) = \frac{\partial}{\partial x_k}\left(\frac{\partial \theta(\boldsymbol{y}, \boldsymbol{z})/\partial y_i}{\partial \theta(\boldsymbol{y}, \boldsymbol{z})/\partial y_j}\right) = 0,\ \forall\ i, j = 1, \cdots, n_y,\ \forall\ k = 1, \cdots, n_x, \tag{A.19}$$

Thus, $\boldsymbol{y}$ is separable from $\boldsymbol{x}$ in the joint production transformation function (Goldman and Uzawa 1964, Lemma 1), that is, $F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \tilde{F}(\boldsymbol{x}, \boldsymbol{z}, \theta(\boldsymbol{y}, \boldsymbol{z}))$. ■

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