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Building Gorman's Nest

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ABSTRACT

The theory of Gorman Engel curves is reviewed and synthesized. A small gap is filled in the set of solutions for this class of indirect preferences. This theory is extended comprehensively to incomplete systems. This extension allows the separate roles of symmetry and adding up to be identified in the rank and functional form restrictions. Symmetry alone determines the rank condition and the maximum rank is three both for incomplete and complete systems. Adding up determines the functional form restrictions in a complete system and there is no restriction on functional form for an incomplete system. We prove that every full rank and minimal deficit reduced rank Gorman system can be written as a polynomial in a single function of income. We use this characterization to obtain a complete taxonomy of closed form solutions for indirect preferences of all of these Gorman systems. We then develop a method to nest the rank and functional form in Gorman systems and present two large classes of demand systems that illustrate the method.

KEY WORDS: Aggregation, functional form, Gorman Engel curves, incomplete demand systems, rank, weak integrability

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1. Introduction

One of the late Terence Gorman's great legacies is his set of seminal contributions to the theory of aggregation in consumption and demand (Gorman 1953, 1961, 1981). Gorman (1953) first derived the necessary and sufficient conditions for the existence of a representative consumer. He then obtained the indirect preference functions for this class of demand models (Gorman 1961), since known as the *Gorman polar form*. Muellbauer (1975, 1976) extended this to include a nonlinear function of income, obtaining the *price independent generalized linear* (PIGL) and *price independent generalized logarithmic* (PIGLOG) systems. Gorman (1981) soon extended these results greatly by deriving the class of all complete demand systems that can be written as a finite sum of additive functions of nominal income, with each function multiplied by a vector of price functions.

The *rank* of a system of Gorman Engel curves is the number of linearly independent columns in the matrix of price functions that premultiply the income functions. Every complete system of *Gorman Engel curves* satisfies two conditions. First, the rank of the system is at most three. Second, if the rank of the system is at least three, then the income functions that are not constant – one of them must be the constant function due to adding up when budget shares are the left-hand-side variables – are all either real powers of income, integer powers of log–income, or pairs of sine and cosine functions of log–income. Gorman's work in this area forms the foundation of a large and important literature on the theory of exact aggregation in demand (Deaton and Muellbauer 1980; Jerison 1993; Lewbell 1987, 1988, 1989, 1990; Muellbauer 1975, 1976; Russell 1983, 1996; Russell and Farris 1993, 1998; and van Daal and Merkies 1989).

This paper extends the theory of Gorman Engel curves to incomplete systems. The incomplete systems approach has enormous potential to expand the way that we think about and successfully model consumption. We show that this approach dramatically increases the set of economically rational Gorman Engel curves and that this extension does not impose additional structure on the goods that are not formally modeled.

The rest of the paper is organized as follows. We first review, synthesize, and extend the existing results for complete Gorman systems. This material is presented in the next section. Then, because the theory of incomplete demand systems appears to be less well known and understood, the third section briefly reviews this topic and provides a carefully constructed example to motivate an interest in incomplete systems and to illustrate some important differences between incomplete and complete systems.

Section four contains a definition of a class of incomplete Gorman systems that has the greatest possible flexibility both with respect to the subset of goods that is being modeled and the subset that is not, while retaining the fundamental structure introduced by Gorman. We completely characterize the indirect preferences for this class of demand models. In doing so, we are able to isolate the role of symmetry from that of adding up in determining the rank and functional form of the demand equations. We find that symmetry alone determines the rank restriction and consequently the maximum rank is three both for incomplete and complete systems. On the other hand, adding up (or homogeneity) determines the functional form restrictions and there is no restriction on the functional form of an incomplete system. We show that all full rank Gorman systems and a generic class of reduced rank systems can be written as a simple polynomial in just one function of income. Remarkably, this characterization result allows us to construct a complete taxonomy of closed form solutions for the indirect preferences of every possible member of this class of demand models.

In section five we apply our taxonomy of indirect preferences to develop a simple empirical method to nest both the rank and the functional form of almost all Gorman systems. We then derive two large classes of incomplete demand models that illustrate the application of this method. The last section summarizes our results, points out some of the natural implications of these results, and briefly discusses some of our recent empirical experience in applying the nesting procedure. Proofs of the main results and most derivations are contained in the Appendix.

2. Complete Gorman Systems

This section reviews and extends slightly the literature on Gorman Engel curves. The results presented here are the combination and synthesis of several at times quite different approaches. We apply some arguments from differential geometry originally developed by Lie (1880; translated with commentary in Hermann 1975), and make extensive use of Muellbauer (1975, 1976), Gorman (1981), van Daal and Merkies (1989), Lewbel (1987, 1989, 1990), and Russell and Farris (1993, 1998). We present this material in a straightforward and direct manner, using the notation of classical calculus, in an attempt to make the material we have used from differential topology a little more accessible.

We begin with a few definitions and some notation. Let $\boldsymbol{P} \in \mathcal{P} \subset \mathbb{R}_{+}^{n}$ be the vector of market prices for the consumption goods $\boldsymbol{q} \in \mathcal{Q} \subset \mathbb{R}_{+}^{n}$, let $M \in \mathcal{M} \subset \mathbb{R}_{+}$ be total expenditure on consumption goods, and let the consumer's utility function be $u(\boldsymbol{q})$, where

$u:\mathcal{Q}\rightarrow\mathcal{U}\subset\mathbb{R}$ is smooth, increasing, and strictly quasiconcave on $\mathcal{Q}$. We abuse language somewhat and use the sobriquet *income* to denote M throughout. Define the *nominal expenditure function* by

$$E(\boldsymbol{P},u)\equiv\min\left\{\boldsymbol{P}^{\top}\boldsymbol{q}:u(\boldsymbol{q})\geq u\right\}. \tag{1}$$

We assume that $E:\mathcal{P}\times\mathcal{U}\rightarrow\mathcal{M}$ is smooth ($E\in\mathcal{C}^{\infty}$), increasing, 1° homogeneous, and concave in $\boldsymbol{P}$, and increasing in u. We also assume an interior solution for $\boldsymbol{q}$. Thus, symmetry is the essential mathematical property of interest.[1] A complete system of Gorman Engel curves can be defined as the partial differential equations,

$$\boldsymbol{q}=\frac{\partial E(\boldsymbol{P},u)}{\partial\boldsymbol{P}}=\sum_{k=1}^{K}\beta_k(\boldsymbol{P})H_k\left(E(\boldsymbol{P},u)\right), \tag{2}$$

where $\beta_k:\mathcal{P}\rightarrow\mathbb{R}^n$ and $H_k:\mathcal{M}\rightarrow\mathbb{R}$, $k=1,\cdots,K$ are smooth functions of prices and income, respectively. There are several reasons to consider demand systems in this class. But the most common relates to exact aggregation across incomes of individual consumers to market-level demands. Let the density function for the income distribution be $\varphi:\mathcal{M}\rightarrow\mathbb{R}_+$. Then (2) clearly implies that we only need to calculate a total of K cross-sectional moments of the form $\int_{\mathcal{M}}H_k(x)\varphi(x)dx$ to obtain the aggregate demands with average consumption on the left-hand-side. In effect, the question raised by Gorman is, "What restriction does economic theory – homogeneity, adding up, and Slutsky symmetry – place on the functional form and number of the income terms that appear on the right-hand-side of (2)?"

However, demand models often are expressed somewhat differently than (2) in the sense that quantities, prices and income are not necessarily the variables of primary interest. For example, Gorman (1981) takes the natural logarithms of prices and income with budget shares on the left-hand-side. Lewbel (1987, 1989, 1990) and van Daal and Merkies (1989) work directly with (2). In contrast, the methods of Russell (1996) and Russell and Farris (1993, 1998) do not depend explicitly on the coordinate system that is chosen to represent the influences of prices and income on the quantities demanded. It also is common for empirical demand models to be estimated with expenditures as the left-hand-side variables. We therefore need what would appear to be a more general statement of Gorman's question.

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[1] However, we also consider curvature in Lemma 1 below, section five, and the Appendix.

Let $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{P})$, $\boldsymbol{g}:\mathcal{P} \to \mathcal{X} \subset \mathbb{R}^n$, transform $\boldsymbol{P}$ to $\boldsymbol{x}$, with $g_i \in \mathcal{C}^\infty$, $i = 1,\cdots,n$, and $\left|\partial \boldsymbol{g}(\boldsymbol{P})^\top / \partial \boldsymbol{P}\right| \neq 0 \ \forall \ \boldsymbol{P} \in \mathcal{P} \subset \mathbb{R}_+^{\boldsymbol{n}}$. Also let $y = f(M)$, $f:\mathcal{M} \to \mathbb{R}$, transform M to y, with $f \in \mathcal{C}^\infty$ and $f'(M) > 0 \ \forall \ M \in \mathcal{M} \subset \mathbb{R}_+$. To simplify notation, we write the inverse of $\boldsymbol{g}$ as $\boldsymbol{P}(\boldsymbol{x})$ and the inverse of f as $M(y)$. Then rather than (2), we might write a system of Gorman Engel curves in terms of the variables $\boldsymbol{x}$ and y as

$$\frac{\partial y(\boldsymbol{x},u)}{\partial \boldsymbol{x}} = \sum_{k=1}^{K} \alpha_k(\boldsymbol{x}) h_k\left(y(\boldsymbol{x},u)\right). \tag{3}$$

But the two definitions are equivalent; by construction, $y(\boldsymbol{x},u) \equiv f\left(E(\boldsymbol{P}(\boldsymbol{x}),u)\right)$, so that

$$\begin{aligned}\frac{\partial y(\boldsymbol{x},u)}{\partial \boldsymbol{x}} &= f'\left(E(\boldsymbol{P}(\boldsymbol{x}),u)\right)\frac{\partial \boldsymbol{P}(\boldsymbol{x})^\top}{\partial \boldsymbol{x}}\frac{\partial E(\boldsymbol{P}(\boldsymbol{x}),u)}{\partial \boldsymbol{P}} \\ &= \sum_{k=1}^{K}\frac{\partial \boldsymbol{P}(\boldsymbol{x})^\top}{\partial \boldsymbol{x}}\beta_k\left(\boldsymbol{P}(\boldsymbol{x})\right)\frac{H_k\left(M\left(y(\boldsymbol{x},u)\right)\right)}{M'\left(M\left(y(\boldsymbol{x},u)\right)\right)} \\ &\equiv \sum_{k=1}^{K}\alpha_k(\boldsymbol{x})h_k\left(y(\boldsymbol{x},u)\right). \end{aligned} \tag{4}$$

Each of these steps is completely reversible. In other words, the functional separability of E from $\boldsymbol{P}$ in the demand system (2) is completely equivalent to functional separability of y from $\boldsymbol{x}$ in the transformed demand system (3). As a consequence, the mathematical structure of a system of Gorman Engel curves is independent of the coordinate space that we might choose to reflect how prices and income influence consumption choices. The following simple and intuitively appealing lemma proves to be very useful in the arguments presented below. This result lets us freely move from one representation of $(y, \boldsymbol{x})$ to another whenever this is convenient without any need to reconsider the implication for integrability. Specifically, the lemma shows that symmetry is (trivially) independent of coordinates. In contrast, additional structure is required to maintain the appropriate curvature of the expenditure function.

Lemma 1. *If the expenditure function is twice differentiable on* $\mathcal{P} \times \mathcal{U} \subset \mathbb{R}_+^n \times \mathbb{R}$, $y = f(E)$, $f \in \mathcal{C}^2$, $f' > 0 \ \forall \ M \in \mathcal{M}$, $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{P})$, $\boldsymbol{g} \in \mathcal{C}^2 \ \forall \ \boldsymbol{p} \in \mathcal{P}$, *and the expenditure function satisfies* $E(\boldsymbol{P},u) = M\left(y(\boldsymbol{g}(\boldsymbol{P}),u)\right)$, *then* $\partial^2 E(\boldsymbol{P},u) / \partial \boldsymbol{P} \partial \boldsymbol{P}^\top$ *is symmetric at* ($\boldsymbol{P}$,u) *if and only if* $\partial^2 y(\boldsymbol{g}(\boldsymbol{P}),u) / \partial \boldsymbol{x} \partial \boldsymbol{x}^\top$ *is symmetric at* ($\boldsymbol{g}(\boldsymbol{P})$,$u$). *If* $x_i = g_i(P_i), g_i \in \mathcal{C}^2, g_i' > 0, g_i'' \le 0, \forall \ p_i \in \mathcal{P}_i \subset \mathbb{R}_+, \forall \ i$, $M'' \le 0 \ \forall \ y \in \mathcal{Y}$ *and* y *is concave in* $\boldsymbol{x}$, *then* E *is concave in* $\boldsymbol{P}$.

We need two conditions on the number of goods relative to the number of income functions and the relationship between and among the price and income functions to ensure that the demand system has a unique representation. Let the $n\times K$ matrix of price functions be denoted by $\boldsymbol{A}(\boldsymbol{x}) = [\alpha_1(\boldsymbol{x})\cdots\alpha_K(\boldsymbol{x})]$ and let the $K\times 1$ vector of income functions be denoted by $\boldsymbol{h}(y)$. The first condition we need is that the $\{h_k(y)\}_{k=1}^K$ are linearly independent with respect to the constants in K–dimensional space. That is, there can exist no $\boldsymbol{c} \in \mathbb{R}^K$ satisfying $\boldsymbol{c} \neq \boldsymbol{0}$ and $\boldsymbol{c}^\top \boldsymbol{h}(y^1) = 0 \;\; \forall\; y^1 \in \mathcal{N}(y) \subset \mathbb{R}$, where $\mathcal{N}(y)$ is an open neighborhood of an arbitrary point in the interior of $\mathcal{Y} \subset \mathbb{R}$, the domain of definition for y. The reason we need this condition is that if it is not satisfied, then $\forall\; \boldsymbol{d} \in \mathbb{R}^K$, adding the n–vector $\boldsymbol{A}(\boldsymbol{x})\boldsymbol{d}\boldsymbol{c}^\top \boldsymbol{h}(y) \equiv \boldsymbol{0}$ to the system of demands does not change it,

$$\begin{aligned}\frac{\partial y}{\partial \boldsymbol{x}} &= \sum_{k=1}^{K} \alpha_k(\boldsymbol{x}) h_k(y) + \sum_{k=1}^{K} \alpha_k(\boldsymbol{x}) d_k \left(\sum_{\ell=1}^{K} c_\ell h_\ell(y) \right) \\ &= \sum_{k=1}^{K} \alpha_k(\boldsymbol{x}) \Big[h_k(y) + d_k \textstyle\sum_{\ell=1}^{K} c_\ell h_\ell(y) \Big] \\ &= \boldsymbol{A}(\boldsymbol{x}) \big(\boldsymbol{I} + \boldsymbol{d}\boldsymbol{c}^\top \big) \boldsymbol{h}(y). \end{aligned} \tag{5}$$

We could therefore choose different $\boldsymbol{d}$ vectors to make the matrix $\boldsymbol{A}(\boldsymbol{x})$ become anything at all, and the demand model would be meaningless.

Similarly, we require that the columns of $\boldsymbol{A}(\boldsymbol{x})$ are linearly independent with respect to the K–dimensional constants. For this to hold, there can be no $\boldsymbol{c} \in \mathbb{R}^K$ that satisfies $\boldsymbol{c} \neq \boldsymbol{0}$ and $\boldsymbol{A}(\boldsymbol{x}^1)\boldsymbol{c} = \boldsymbol{0} \;\; \forall\; \boldsymbol{x}^1 \in \mathcal{N}(\boldsymbol{x})$, where here $\mathcal{N}(\boldsymbol{x})$ is an open neighborhood of any point in the interior of $\mathcal{X} \subset \mathbb{R}^n$, the domain of definition for $\boldsymbol{x}$. As before, if this did not hold, then $\forall\; \boldsymbol{d} \in \mathbb{R}^K$, adding $\boldsymbol{A}(\boldsymbol{x})\boldsymbol{c}\boldsymbol{d}^\top \boldsymbol{h}(y) \equiv \boldsymbol{0}$ to the system does not change it,

$$\begin{aligned}\frac{\partial y}{\partial \boldsymbol{x}} &= \sum_{k=1}^{K} \Big[\alpha_k(\boldsymbol{x}) + \Big(\textstyle\sum_{\ell=1}^{K} \alpha_\ell(\boldsymbol{x}) c_\ell \Big) d_k \Big] h_k(y) \\ &= \boldsymbol{A}(\boldsymbol{x}) \big(\boldsymbol{I} + \boldsymbol{c}\boldsymbol{d}^\top \big) \boldsymbol{h}(y). \end{aligned} \tag{6}$$

We could again choose different vectors $\boldsymbol{d}$ to make the matrix $\boldsymbol{A}(\boldsymbol{x})$ become anything, and the demand model makes no sense. We assume throughout that the dimensions of $\boldsymbol{A}$ and $\boldsymbol{h}$ have been reduced as necessary to guarantee a unique representation (see Gorman 1981: 358-59; or Russell and Farris 1998: 201-202).[2]

[2] It is important to emphasize that linear independence across the K–dimensional constants is not equivalent to $\boldsymbol{A}(\boldsymbol{x})$ having full column rank. In particular, if we let the vector of coefficients in a linear combination of the columns of $\boldsymbol{A}(\boldsymbol{x})$ be functions of $\boldsymbol{x}$ and/or y, a vector $\boldsymbol{c}(\boldsymbol{x},y)$ may exist that satisfies $\boldsymbol{A}(\boldsymbol{x})\boldsymbol{c}(\boldsymbol{x},y) = 0$ even if both of the required properties discussed immediately above are satisfied.

We are now ready to proceed with the analysis. Gorman (1981) proved that all complete demand systems with the structure of (3) must have a rank of $\boldsymbol{A}(\boldsymbol{x})$ that is at most equal to three. If the rank of $\boldsymbol{A}(\boldsymbol{x})$ is at least three, then the system must take one of the following three functional forms:

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})M + \sum_{k=1}^{K} \alpha_k(\boldsymbol{x})M(\ln M)^k \; ; \tag{7}$$

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})M + \sum_{\kappa \in S} \beta_\kappa(\boldsymbol{x})M^{1-\kappa} + \sum_{\kappa \in S} \gamma_\kappa(\boldsymbol{x})M^{1+\kappa} \, , \tag{8}$$

for S a set of nonzero constants; or

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})M + \sum_{\tau \in T} \beta_\tau(\boldsymbol{x})M \sin\left(\tau \ln M\right) + \sum_{\tau \in T} \gamma_\tau(\boldsymbol{x})M \cos\left(\tau \ln M\right), \tag{9}$$

for T a set of positive constants. This includes PIGLOG models and extensions of it that are polynomials in $\ln(M)$, simple polynomials in income, and PIGL models and extensions of it with power functions of the form M^κ, in addition to the trigonometric form (9). Gorman's theorem completely identifies the set of all possible functional forms for rank three Gorman systems, although the total number of possible different income terms is not addressed by this result.

A Gorman system has *full rank* (Lewbel 1990) if the rank of $\boldsymbol{A}(\boldsymbol{x})$ is equal to the number of columns and therefore to the number of income functions, $h_k(y)$. We know a great deal about the class of all full rank Gorman systems. All full rank one complete systems are homothetic,

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})M \; , \tag{10}$$

due to adding up. In budget share form, therefore, any full rank one complete system must be a zero–order polynomial in income. Muellbauer (1975, 1976) proved that all full rank two complete systems are either PIGL or PIGLOG; either

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})M + \alpha_1(\boldsymbol{x})M^{1-\kappa} . \tag{11}$$

for some $\kappa \neq 0$, or

$$\boldsymbol{q} = \alpha_0(\boldsymbol{x})M + \alpha_1(\boldsymbol{x})M \ln M \; . \tag{12}$$

Here is one simple way to see the reason for this result. This approach also is useful for understanding higher rank cases. A first-order ordinary differential equation is said to be a *Bernoulli equation* if it can be written in the form,

$$\frac{d[y(x)^{\kappa}]}{dx} = \kappa y(x)^{\kappa-1} y'(x) = \beta_0(x) + \beta_1(x) y(x)^{\kappa}. \tag{13}$$

The solution to this linear, first-order, ordinary differential equation is found by applying the integrating factor $e^{-\int^{x} \beta_1(s)ds}$, since

$$\frac{d}{dx}\left[y(x)^{\kappa}\, e^{-\int^{x} \beta_1(s)ds} \right] = \left[\kappa y(x)^{\kappa-1} y'(x) - \beta_1(x) y(x)^{\kappa} \right] e^{-\int^{x} \beta_1(s)ds}. \tag{14}$$

We can extend this argument to systems of integrable partial differential equations. We say that a system of first-order partial differential equations is a *linear Bernoulli system* if it can be written in the form (note that the utility index plays no role in the definition),

$$\frac{\partial[E(\boldsymbol{P},u)^{\kappa}]}{\partial \boldsymbol{P}} = \kappa E(\boldsymbol{P},u)^{\kappa-1} \frac{\partial E(\boldsymbol{P},u)}{\partial \boldsymbol{P}} = \beta_0(\boldsymbol{P}) + \beta_1(\boldsymbol{P}) E(\boldsymbol{P},u)^{\kappa}. \tag{15}$$

If we multiply this expression by $\kappa^{-1}E(\boldsymbol{P},u)^{\kappa-1}$, then we obtain the PIGL form,

$$\frac{\partial E(\boldsymbol{P},u)}{\partial \boldsymbol{P}} = \alpha_0(\boldsymbol{P}) E(\boldsymbol{P},u) + \alpha_1(\boldsymbol{P}) E(\boldsymbol{P},u)^{1-\kappa}, \tag{16}$$

with an obvious redefinition of the price functions.

A similar analysis applies to the logarithmic functional form. A first-order ordinary differential equation is said to be a *logarithmic transformation* if it has the form

$$\frac{d[\ln y(x)]}{dx} = \frac{y'(x)}{y(x)} = \beta_0(x) + \beta_1(x) \ln y(x). \tag{17}$$

This also is a linear, first-order, ordinary differential equation in $\ln y$ and the same integrating factor can be used to obtain its solution. Again, we can extend this to systems of partial differential equations. We say that a system of first-order partial differential equations is a *linear logarithmic system* if it has the form

$$\frac{\partial \ln\left[E(\boldsymbol{P},u)\right]}{\partial \boldsymbol{P}} = \frac{\partial E(\boldsymbol{P},u)/\partial \boldsymbol{P}}{E(\boldsymbol{P},u)} = \alpha_0(\boldsymbol{P}) + \alpha_1(\boldsymbol{P}) \ln\left[E(\boldsymbol{P},u)\right]. \tag{18}$$

In this case, multiplying this expression by $E(\boldsymbol{P},u)$ gives the PIGLOG form

$$\frac{\partial E(\boldsymbol{P},u)}{\partial \boldsymbol{P}} = \alpha_0(\boldsymbol{P}) E(\boldsymbol{P},u) + \alpha_1(\boldsymbol{P}) E(\boldsymbol{P},u) \ln\left[E(\boldsymbol{P},u)\right]. \tag{19}$$

Hence, all full rank two complete systems are first-order polynomials in a single function of income.

These two functional forms combined with the adding up conditions $\alpha_0(\boldsymbol{P})^{\mathsf{T}}\boldsymbol{P}\equiv 1$ and $\alpha_1(\boldsymbol{P})^{\mathsf{T}}\boldsymbol{P}\equiv 0$ are the only possible ways that full rank two systems can satisfy zero degree homogeneity in prices and income and the budget identity. One straightforward way to see how the restriction on functional form follows from the adding up condition is the following.[3] Let the transformed expenditure function be $y(\boldsymbol{P},u)=f\left(E(\boldsymbol{P},u)\right)$, so that in the rank two case with $\boldsymbol{x}=\boldsymbol{P}$, the system (4) has the form,

$$f'(M)\boldsymbol{q}=\alpha_0(\boldsymbol{P})H_0\left(f(M)\right)+\alpha_1(\boldsymbol{P})H_1\left(f(M)\right). \tag{20}$$

Now if we apply the budget identity, we have

$$f'(M)M=\boldsymbol{P}^{\mathsf{T}}\alpha_0(\boldsymbol{P})H_0(f(M))+\boldsymbol{P}^{\mathsf{T}}\alpha_1(\boldsymbol{P})H_1(f(M)). \tag{21}$$

Since the left-hand-side only depends on *M*, so must the right-hand-side. On the other hand, since the right-hand-side can only be a function of $y=f(M)$, so must the left-hand-side. Consistent aggregation for complete systems, therefore, is defined by the class of functional forms for *f* that satisfy $f'(M)M=a+bf(M)$ for some constants *a* and *b*. The reason for this condition is that it is necessary and sufficient for both the left-hand-side and the right-hand-side of (21) to always aggregate across individuals to the same average value.[4] Integrating this simple linear first-order ordinary differential equation implies the following simple result for the class of possible functional forms.

Lemma 2. *$f:\mathbb{R}\to\mathbb{R}$, $f\in\mathcal{C}^{\infty}$, $f'>0$, satisfies the linear first-order ordinary differential equation $Mf'(M)=a+bf(M)\ \forall\ M\in\mathcal{M}$ if and only if*

$$f(M)=\begin{cases} a\ln M+c, & \text{if } b=0,\\ \\ cM^{b}-(a/b), & \text{if } b>0.\end{cases}$$

For demand models, without loss in generality we can set $a = 1$, $c = 0$ if $b = 0$, and $c = 1$ if $b > 0$, because the functions α_0 and α_1 can absorb other values for *a* and/or *c*. If we rename *b* by λ, then the set of Box-Cox transformations of income,

$$f(M)=(M^{\lambda}-1)/\lambda,\ \lambda\geq 0, \tag{22}$$

completely characterizes this class of demand systems. Note that $f'(M)=M^{\lambda-1}$, so that

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[3] Also, see Muellbauer (1975, 1976) for a detailed discussion and in depth analysis.

[4] Indeed, this is a precise interpretation of the term *generalized linearity* in Muellbauer's definition of the PIGL and PIGLOG functional forms.

$Mf'(M) = M^{\lambda} = 1 + \lambda f(M)$. This implies that one income function, say H_0, must be the constant function, while the other, say H_1, must be $f(M)$, and hence the demand system can be written as

$$M^{\lambda-1}\boldsymbol{q} = \alpha_0(\boldsymbol{P}) + \alpha_1(\boldsymbol{P})(M^{\lambda} - 1)/\lambda . \quad (23)$$

Applying adding up to the right-hand-side then implies that

$$\boldsymbol{P}^{\top}\alpha_0(\boldsymbol{P}) + \boldsymbol{P}^{\top}\alpha_1(\boldsymbol{P})(M^{\lambda} - 1)/\lambda = 1 + \lambda f(M) , \quad (24)$$

if and only if $\boldsymbol{P}^{\top}\alpha_0(\boldsymbol{P}) = 1$ and $\boldsymbol{P}^{\top}\alpha_1(\boldsymbol{P}) = \lambda$. Clearly, (23) is equivalent to a nested set of all PIGL and PIGLOG demand systems, including the Gorman polar form (Gorman 1953, 1961) and the Almost Ideal Demand System (Deaton and Muellbauer 1980) as two special cases. Note, in particular, that we identified this solution for the full rank two case without appealing to symmetry, only to adding up. The Appendix gives a different derivation of the functional form restriction in the full rank two case that is based solely on homogeneity and therefore also does not appeal to symmetry.

Turning now to the full rank three case, one important implication of Gorman's (1981) method of proof is that if $\boldsymbol{A}(\boldsymbol{x})$ has full rank three, then $K = 2$ in (7), S has one element, κ, which appears once with a negative sign and once with a positive sign in the exponents in (8), and T has one element, τ, appearing once in a sine function and once in a cosine function in (9), forming a complex conjugate pair. Lewbel (1990) used these properties to obtain the full rank three PIGL, PIGLOG, and trigonometric solutions, by applying the symmetry analysis of van Daal and Merkies (1989) to these functional forms. Lewbel (1990) shows that every full rank three complete Gorman system must be one of the following:

(a) a *generalized **P**IGL* (including the QES with κ = 1),

$$\boldsymbol{q} = \alpha_0(\boldsymbol{P})M + \alpha_1(\boldsymbol{P})M^{1-\kappa} + \alpha_2(\boldsymbol{P})M^{1+\kappa} ; \quad (25)$$

(b) a *generalized **P**IGLOG*,

$$\boldsymbol{q} = \alpha_0(\boldsymbol{P})M + \alpha_1(\boldsymbol{P})M \ln M + \alpha_2(\boldsymbol{P})M(\ln M)^2 ; \quad (26)$$

or (c) a trigonometric demand system,

$$\boldsymbol{q} = \alpha_0(\boldsymbol{P})M + \alpha_1(\boldsymbol{P})M \sin(\tau \ln M) + \alpha_2(\boldsymbol{P})M \cos(\tau \ln M) . \quad (27)$$

All of the full rank cases can be combined within a single unifying framework. Recall that the full rank one and full rank two cases are zero and first-order polynomials, respectively, in a single function of income. We now show how to represent each full rank three case as a quadratic polynomial in a single associated function of income. First, we extend linear Bernoulli systems by saying that a system of first-order partial differential equations is a *quadratic Bernoulli system* if it can be written in the form,

$$\frac{\partial[E(\boldsymbol{P},u)^{\kappa}]}{\partial \boldsymbol{P}} = \kappa E(\boldsymbol{P},u)^{\kappa-1}\frac{\partial E(\boldsymbol{P},u)}{\partial \boldsymbol{P}} = \beta_0(\boldsymbol{P}) + \beta_1(\boldsymbol{P})E(\boldsymbol{P},u)^{\kappa} + \beta_2(\boldsymbol{P})E(\boldsymbol{P},u)^{2\kappa} \quad (28)$$

Multiplying this system by $\kappa^{-1}E(\boldsymbol{P},u)^{1-\kappa}$ generates the full rank three generalized PIGL model, with an obvious rescaling of the vectors of price functions. Similarly, we extend linear logarithmic systems by saying that a system of first–order partial differential equations is a *quadratic logarithmic system* if it can be written in the form,

$$\frac{\partial[\ln E(\boldsymbol{P},u)]}{\partial \boldsymbol{P}} = \frac{\partial E(\boldsymbol{P},u)/\partial \boldsymbol{P}}{E(\boldsymbol{P},u)} = \alpha_0(\boldsymbol{P}) + \alpha_1(\boldsymbol{P})\ln E(\boldsymbol{P},u) + \alpha_2(\boldsymbol{P})\left[\ln E(\boldsymbol{P},u)\right]^2. \quad (29)$$

Multiplying this expression by $E(\boldsymbol{P},u)$ gives the full rank three generalized PIGLOG. Finally, we define a *quadratic complex exponential system* of first–order partial differential equations as follows:

$$\frac{\partial[(\iota\tau)^{-1}E(\boldsymbol{P},u)^{\iota\tau}]}{\partial \boldsymbol{x}} = E(\boldsymbol{P},u)^{\iota\tau-1}\frac{\partial E(\boldsymbol{P},u)}{\partial \boldsymbol{P}}$$

$$= \tfrac{1}{2}\left[\alpha_1(\boldsymbol{P}) + \iota\alpha_2(\boldsymbol{P})\right] + \alpha_0(\boldsymbol{P})E(\boldsymbol{P},u)^{\iota\tau} + \tfrac{1}{2}\left[\alpha_1(\boldsymbol{P}) - \iota\alpha_2(\boldsymbol{P})\right]E(\boldsymbol{P},u)^{2\iota\tau}, \quad (30)$$

with $\iota = \sqrt{-1}$. Multiplying this system by $E(\boldsymbol{P},u)^{1-\iota\tau}$ and invoking de Moivre's theorem,

$$e^{\pm\iota\tau y} = \cos(\tau y) \pm \iota\sin(\tau y), \quad (31)$$

with $y = \ln M$, produces the trigonometric demand system in case (c) above.[5] The main trick in this case is to carefully select the correct pair of complex conjugate combinations of the price functions α_0, α_1, and α_2 in the original (complex-valued) quadratic form to obtain real-valued demands. In all other respects, the steps are the same as for the generalized PIGL and PIGLOG cases.

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[5]Including $(\iota\tau)^{-1}$ on the left of (30) is innocuous, since $1/\iota = -\iota$. The right-hand-side can be multiplied by $\iota\tau$ and this complex constant can then be absorbed into the complex conjugate price vectors without changing the structure of the final result. We chose this form for (30) because it makes clear the fact that the price functions for $M^0 = 1$ and $M^{2\iota\tau}$ must be complex conjugates, while the price function for $M^{\iota\tau}$ must be real in order that both E and $\boldsymbol{q}$ are real-valued functions.

All full rank complete Gorman Engel curve systems can be represented as a zero-, first-, or second-order polynomial of only one function of income.

This conclusion was recognized in a series of deep and, unfortunately, what appear to be considerably less well known and understood papers by Russell (1983, 1996) and Russell and Farris (1993, 1998).[6] These articles establish the connection between Gorman systems and the theory of Lie transformation groups (Hermann 1975). Russell (1983) initially argued that Gorman's theorem follows from Lie's result on the maximal rank of local transformation groups on the real line. But Jerison (1993) presented a counterexample to this claim based on a polynomial demand system with more than three income functions (and therefore reduced rank) that is not a local Lie transformation group. However, Russell and Farris (1993) apply Lie's theory to show that every full rank Gorman system can be written as a special case of the quadratic system

$$f'(M)\boldsymbol{q} = \alpha_0(\boldsymbol{P}) + \alpha_1(\boldsymbol{P})f(M) + \alpha_2(\boldsymbol{P})f(M)^2, \tag{32}$$

for some smooth, strictly increasing function $y = f(M)$. The essence of why this must be true is captured in the derivations leading to (28)–(30) above.

Russell and Farris (1993) also show that adding up restricts the functional form to the cases identified by Gorman (1981). Using the same device as for the rank two case, we can apply the adding up condition to (32) to obtain,

$$\begin{aligned} Mf'(M) &= \alpha_0(\boldsymbol{P})^\top \boldsymbol{P} + \alpha_1(\boldsymbol{P})^\top \boldsymbol{P}f(M) + \alpha_2(\boldsymbol{P})^\top \boldsymbol{P}f(M)^2 \\ &= a + bf(M) + cf(M)^2, \end{aligned} \tag{33}$$

for some absolute constants, *a*, *b*, *c*. Since $\{1 \;\, f(M) \;\, f(M)^2\}$ are linearly independent, we must have (a) $Mf'(m) = a$; (b) $Mf'(M) = bf(M)$, or (c) $Mf'(M) = cf(M)^2$. The first two cases are the same as for the rank two case, with an added possibility that *b* can be either purely real or purely complex. Case (c) implies $f(M) = -1/\ln M$. But this duplicates case (a) if we introduce the monotone transformation $-1/f(M) = \ln M$, without any loss in generality. We also can combine cases (a) and (b) when $b \geq 0$ is real using a Box-Cox transformation on income to nest this subset of full rank three systems.

Russell and Farris (1998) extend these results to show that Jerison's counterex-

[6] We are indebted to Thomas Russell for pointing us in the direction of differential geometry as a strategy for addressing the structure of incomplete Gorman systems.

ample is the only one possible in an important and generic sense (also see Russell 1996).[7] That is, if there are $K \geq 3$ linearly independent income functions and a maximal number of terms of the form, $h_k(y)h'_\ell(y) - h'_k(y)h_\ell(y),\ k < \ell$, can be written as exact linear combinations of the functions $\{h_k(y)\}_{k=1}^{K}$,

$$h_k(y)h'_\ell(y) - h'_k(y)h_\ell(y) = c^1_{k\ell}h_1(y) + c^2_{k\ell}h_2(y) + \cdots c^K_{k\ell}h_K(y)\,, \tag{34}$$

with the $c^j_{k\ell},\ j = 1,\cdots,K,\ k < \ell$, a set of absolute constants (that is, they do not depend on $\boldsymbol{P}$ or y), then a representation (definition for $y = f(M)$) exists such that the demand system can be written in the polynomial form[8]

$$f'(M)\boldsymbol{q} = \alpha_1(\boldsymbol{P}) + \alpha_2(\boldsymbol{P})f(M) + \alpha_3(\boldsymbol{P})f(M)^2 + \cdots + \alpha_K(\boldsymbol{P})f(M)^{K-1}. \tag{35}$$

Of course, the functional form restrictions found by Gorman (1981) continue to apply in all cases with $\operatorname{rank}\left(\boldsymbol{A}(\boldsymbol{x})\right) = 3 \leq K$. Russell and Farris (1998) also show that the theory of Lie algebras on the real line implies that the rank of Gorman systems is at most three, but remark in a footnote that Gorman's (1981) theorem was a nontrivial extension of this area of differential geometry.

We conclude this section by filling a gap in the literature on full rank three cases. Applying minor modifications to their arguments, it is possible to show that van Daal and Merkies (1989) and Lewbel (1990) proved that every full rank three QES, PIGL, and PIGLOG system will be integrable if and only if four functions, $\beta_1, \beta_2, \beta_3 : \mathscr{P} \to \mathbb{R}$, and $\gamma : \mathbb{R} \to \mathbb{R}$, exist such that

$$\frac{\partial y(\boldsymbol{P},u)}{\partial \boldsymbol{P}} = \frac{\partial \beta_1(\boldsymbol{P})}{\partial \boldsymbol{P}} + \gamma\left(\beta_2(\boldsymbol{P})\right)\beta_3(\boldsymbol{P})\frac{\partial \beta_2(\boldsymbol{P})}{\partial \boldsymbol{P}}$$
$$+\frac{\partial \beta_3(\boldsymbol{P})}{\partial \boldsymbol{P}}\frac{\left[y(\boldsymbol{P},u) - \beta_1(\boldsymbol{P})\right]}{\beta_3(\boldsymbol{P})} + \frac{\partial \beta_2(\boldsymbol{P})}{\partial \boldsymbol{P}}\frac{\left[y(\boldsymbol{P},u) - \beta_1(\boldsymbol{P})\right]^2}{\beta_3(\boldsymbol{P})}, \tag{36}$$

[7] Our use of the word *generic* here is intended to convey our belief that Theorem 4 in Russell and Farris (1998) gives a precise meaning to the intriguing last paragraph and footnote in Gorman (1981).

[8] In differential geometry, the $h_k(y)h'_\ell(y) - h'_k(y)h_\ell(y)$ are *Jacoby brackets* and the absolute constants $\{c^j_{k\ell}\}$ are the *structure constants* of the system of partial differential equations defining the demand model. If we append the differential operator, $\partial/\partial y$ to the right-hand-side of the Jacoby brackets, we obtain the *Lie brackets*, $[h_k(y)h'_\ell(y) - h'_k(y)h_\ell(y)]\partial/\partial y$. The differential operators, $h_k(y)\partial/\partial y$ are vector fields on the real line and the system of equations (34) is a *Lie algebra* on the finite dimensional vector space spanned by these operators. Russell and Farris (1993) contains a helpful introduction to these concepts and their role in Gorman systems. Hydon (2000), Guillemin and Pollack (1974), and Spivak (1999) also are useful sources.

where $y(\boldsymbol{P},u) \equiv f\left(E(\boldsymbol{P},u)\right)$ for an appropriately chosen f in each case. This expression can be rewritten in the form,

$$\frac{\partial}{\partial \boldsymbol{P}}\left(\frac{y(\boldsymbol{P},u)-\beta_1(\boldsymbol{P})}{\beta_3(\boldsymbol{P})}\right)=\frac{1}{\beta_3(\boldsymbol{P})}\left(\frac{\partial y(\boldsymbol{P},u)}{\partial \boldsymbol{P}}-\frac{\partial \beta_1(\boldsymbol{P})}{\partial \boldsymbol{P}}\right)-\frac{\left[y(\boldsymbol{P},u)-\beta_1(\boldsymbol{P})\right]}{\beta_3(\boldsymbol{P})^2}\frac{\partial \beta_3(\boldsymbol{P})}{\partial \boldsymbol{P}}$$

$$=\left[\gamma\left(\beta_2(\boldsymbol{P})\right)+\left(\frac{y(\boldsymbol{P},u)-\beta_1(\boldsymbol{P})}{\beta_3(\boldsymbol{P})}\right)^2\right]\frac{\partial \beta_2(\boldsymbol{P})}{\partial \boldsymbol{P}}. \tag{37}$$

This second expression can in turn be simplified further with two successive changes of variables. First, let $w(\boldsymbol{P},u)=\left[y(\boldsymbol{P},u)-\beta_1(\boldsymbol{P})\right]/\beta_3(\boldsymbol{P})$, so that

$$\frac{\partial w(\boldsymbol{P},u)}{\partial \boldsymbol{P}}=\left[\gamma\left(\beta_2(\boldsymbol{P})\right)+w(\boldsymbol{P},u)^2\right]\frac{\partial \beta_2(\boldsymbol{P})}{\partial \boldsymbol{P}} \tag{38}$$

Second, let $z(\boldsymbol{P},u)=-1/w(\boldsymbol{P},u)$, so that

$$\frac{\partial z(\boldsymbol{P},u)}{\partial \boldsymbol{P}}=\left[1+\gamma\left(\beta_2(\boldsymbol{P})\right)z(\boldsymbol{P},u)^2\right]\frac{\partial \beta_2(\boldsymbol{P})}{\partial \boldsymbol{P}}. \tag{39}$$

Now, if $\gamma\left(\beta_2(\boldsymbol{P})\right)\equiv\lambda$ is a constant, then we can separate the variables in (39),

$$\frac{\partial z(\boldsymbol{P},u)/\partial P_i}{1+\lambda z(\boldsymbol{P},u)^2}=\frac{\partial \beta_2(\boldsymbol{P})}{\partial P_i}\ \forall\ i=1,\cdots,n. \tag{40}$$

This is an *exact* system and its solution can be found by direct integration,

$$\phi\left(\frac{-\beta_3(\boldsymbol{P})}{y(\boldsymbol{P},u)-\beta_1(\boldsymbol{P})}\right)\equiv\int^{-\beta_3(\boldsymbol{P})/\left[y(\boldsymbol{P},u)-\beta_1(\boldsymbol{P})\right]}\frac{dz}{(1+\lambda z^2)}=\beta_2(\boldsymbol{P})+u, \tag{41}$$

where we normalize by setting the utility index equal to the constant of integration. This is the solution reported in van Daal and Merkies (1989) and Lewbel (1987, 1990).

However, we can go somewhat further. First, suppose that $\lambda>0$ and make a third change of variables to $s=\kappa z$, where $\lambda=\kappa^2>0$, so that

$$\frac{1}{\kappa}\tan^{-1}\left(\frac{-\kappa\beta_3(\boldsymbol{P})}{y(\boldsymbol{P},u)-\beta_1(\boldsymbol{P})}\right)=\int^{-\kappa\beta_3(\boldsymbol{P})/\left[y(\boldsymbol{P},u)-\beta_1(\boldsymbol{P})\right]}\frac{ds}{\kappa(1+s^2)}=\beta_2(\boldsymbol{P})+u. \tag{42}$$

Solving for u, the indirect utility function in this case is

$$v(\boldsymbol{P},M)=\frac{1}{\kappa}\tan^{-1}\left(\frac{-\kappa\beta_3(\boldsymbol{P})}{f(M)-\beta_1(\boldsymbol{P})}\right)-\beta_2(\boldsymbol{P}) \tag{43}$$

Conversely, if $\lambda < 0$, define $-\lambda = \kappa^2$ and write $1+\lambda z^2 = (1+\kappa z)(1-\kappa z)$. The method of partial fractions then implies

$$\frac{1}{1+\lambda z^2} = \frac{½}{(1-\kappa z)} + \frac{½}{(1+\kappa z)}, \tag{44}$$

and direct integration gives

$$½\ln\left(\frac{y(\boldsymbol{P},u)-\beta_1(\boldsymbol{P})-\kappa\beta_3(\boldsymbol{P})}{y(\boldsymbol{P},u)-\beta_1(\boldsymbol{P})+\kappa\beta_3(\boldsymbol{P})}\right) = \int^{-\beta_3(\boldsymbol{P})/[y(\boldsymbol{P},u)-\beta_1(\boldsymbol{P})]} \frac{dz}{(1+\lambda z^2)} = \beta_2(\boldsymbol{P}) + ½\ln u\,. \tag{45}$$

In this case, it is most convenient to normalize the constant of integration by ½ln*u*. If we exponentiate both sides, construct a new $\beta_2(\boldsymbol{P})$ from the old one as $e^{2\beta_2(\boldsymbol{P})}$, and solve for *u*, we obtain the indirect utility function for this case in the form,

$$v(\boldsymbol{P},M) = \frac{f(M)-\beta_1(\boldsymbol{P})-\kappa\beta_3(\boldsymbol{P})}{\beta_2(\boldsymbol{P})\left[f(M)-\beta_1(\boldsymbol{P})+\kappa\beta_3(\boldsymbol{P})\right]}. \tag{46}$$

Thus, we have found the closed form expressions for this group of full rank three cases.

Neither Lewbel (1987, 1989, 1990) nor van Daal and Merkies (1989) found a closed form solution for the indirect utility function if $\gamma\left(\beta_3(\boldsymbol{P})\right)$ is not constant. We now show that only the constant λ case generalizes the original solution for the QES obtained by Howe, Pollak, and Wales (1979), and so their quest was already over at this point.

Lemma 3. *If* $z:\mathcal{P}\times\mathcal{U}\to\mathbb{R},\ z\in\mathcal{C}^\infty$ *satisfies the partial differential equations,*

$$\frac{\partial z(\boldsymbol{P},u)}{\partial \boldsymbol{P}} = \left[1+\gamma\left(\beta_2(\boldsymbol{P})\right)z(\boldsymbol{P},u)^2\right]\frac{\partial\beta_2(\boldsymbol{P})}{\partial \boldsymbol{P}},$$

then either $\gamma\left(\beta_2(\boldsymbol{P})\right)\equiv\lambda$ *is constant or*

$$\frac{\partial z(\boldsymbol{P},u)}{\partial \boldsymbol{P}} = \gamma'\left(\beta_2(\boldsymbol{P})\right)\frac{\partial\beta_2(\boldsymbol{P})}{\partial \boldsymbol{P}} \equiv \frac{\partial\gamma\left(\beta_2(\boldsymbol{P})\right)}{\partial \boldsymbol{P}}.$$

Remark: The original solution obtained by Howe, Pollak and Wales (1979) for the QES has the general form (using the notation in van Daal and Merkies),

$$v(\boldsymbol{P},M) = \left(\frac{\beta_3(\boldsymbol{P})}{\beta_1(\boldsymbol{P})-M}\right) - \beta_2(\boldsymbol{P}). \tag{47}$$

Recall that the variable *z* is obtained through a change of variables from *w* to $z=-w^{-1}$, where *w* is the Gorman polar form, $w(\boldsymbol{P},u) = \left[y(\boldsymbol{P},u)-\beta_1(\boldsymbol{P})\right]/\beta_3(\boldsymbol{P})$. As a result, the variables can again be separated if the second case of Lemma 3 holds. Direct integration then gives

$$\frac{-\beta_3(\boldsymbol{P})}{y(\boldsymbol{P},u)-\beta_1(\boldsymbol{P})} = \int^{-\beta_3(\boldsymbol{P})/[y(\boldsymbol{P},u)-\beta_1(\boldsymbol{P})]} dz = \gamma\left(\beta_2(\boldsymbol{P})\right) + u\,. \tag{48}$$

Therefore, if $\gamma'\left(\beta_2(\boldsymbol{P})\right) \neq 0$, redefine $\gamma\left(\beta_2(\boldsymbol{P})\right)$ as $\beta_2(\boldsymbol{P})$ and recall that $f(M) = M$ in the QES. This reproduces the original QES solution in Howe, Pollak, and Wales (1979). Thus, the only relevant case that they missed is $\gamma\left(\beta_2(\boldsymbol{P})\right) = \lambda > 0$ – i.e., complex roots for the *Ricatti system* of partial differential equations in (39) and the inverse tangent indirect utility function in (43) above. The basis for this assertion is that we can easily renormalize the indirect utility function (46) to be written in the form given by (47). ■

We have limited our discussion in this section to complete nominal Gorman systems. However, Lewbel (1989) showed that Gorman systems with deflated income can achieve rank four. This result, and the detailed analysis of the structure of deflated income systems in Russell and Farris (1998), plays an important role in section four, where we turn to incomplete Gorman systems. Indeed, it is precisely Lewbel's (1989) rank four result that permits us to guarantee that we retain the maximum flexibility for both rank and functional form of the subset of goods that are not being modeled when we specify an incomplete Gorman system in the way that we do in that section. We will defer a further discussion of this issue until we get to that point in the paper.

3. Incomplete Demand Systems

In everything reviewed in the previous section, only Russell and Farris (1993) even mention an incomplete system. They argue that (32) above completely characterizes all full rank incomplete Gorman systems for any smooth, strictly increasing function of income (page 319). Yet even this statement about the structure of incomplete Gorman systems turns out to be not entirely correct. [9] The reason is that homogeneity is as important as adding up in the functional form of Gorman Engel curves. Any group of demand equations, whether or not they form a complete system, must be 0° homogeneous in all prices and income. Many – in fact, almost all – smooth and strictly monotone functions cannot be made to be 0° homogeneous in prices and income simply by multiplying by some function of prices. For example, no function $\alpha(\boldsymbol{P})$ exists that can make the product

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[9] Nevertheless, if we ignore both homogeneity and adding up altogether, perhaps because we consider them to be spurious or otherwise irrelevant additional side constraints on a system of demands, then the statement by Russell and Farris (1993) is clearly correct.

$\alpha(\boldsymbol{P})e^{\lambda M}$ become 0° homogeneous in ($\boldsymbol{P}$, M). Consequently, the structure of incomplete systems of Gorman Engel curves is unknown territory. It also seems clear that the properties of incomplete demand systems implied by the theory of consumer choice and revealed by market behavior is not well understood.

This section presents a brief overview the properties of incomplete demand systems. The last part of the section then presents a carefully constructed example to motivate an interest in incomplete systems and to demonstrate some of the important differences between complete and incomplete systems. Primary references for this section are Epstein (1982), LaFrance (1985, 2004), and LaFrance and Hanemann (1989).

3.1 The Structure of Incomplete Demand Systems

Incomplete information is the rule not the exception. Most of the time, we are interested in only a subset of all of the items purchased and used by consumers. We may not be interested in the consumer demands for other commodities, we may not have access to all of the required data, or the complexity and computational time needed to analyze a complete system of demand equations for all goods may be prohibitive. Whatever the reason, it is far more common for economists to model a subset of goods than the entire mix of goods that make up the consumer's expenditure decisions.

Three approaches are commonly used to address the complexity of large demand systems. One is to aggregate across commodities and estimate a complete system of demand equations using the commodity aggregates (e.g., food, clothing, housing, fuel, drink and tobacco, transportation and communication, other goods, and other services, as in Deaton and Muellbauer 1980). The second approach appeals to separability of consumer preferences and estimates a complete system of conditional demands for the goods of interest as functions of the prices for those goods and the total expenditure on that group of goods (e.g., beef, mutton, and other meat, as in Wales and Woodland 1983). The third approach specifies an incomplete system of demand equations as functions of the prices of the goods of interest, the prices of related goods, and total expenditure on all goods, or income (e.g., Burt and Brewer 1971, or Chiccetti, Fisher, and Smith 1976). This approach is commonly referred to as an *incomplete demand system*.

Each approach has its own set of advantages and disadvantages. However, the greatest flexibility by far in modeling a subset of goods can be achieved by pursuing the third approach. Taking this route has three impacts on demand models. First, the budget constraint becomes a strict inequality. Second, the demands are not 0° homogeneous in the prices of the goods being modeled and income. Third, there is no way to know if the demand functions for the goods that are not being modeled have the same structure as those that are. It turns out, however, that each of these increases, rather than decreases, the flexibility we have in conducting rational, coherent economic analyses.

To better understand the structure of incomplete demand systems, we first need to redefine some of our previous terms and add a couple of new definitions. We now consider $\boldsymbol{q} \in \boldsymbol{\mathcal{Q}} \subset \mathbb{R}_+^{n_q}$ to be the market goods of primary interest, with nominal market prices $\boldsymbol{P} \in \boldsymbol{\mathcal{P}} \subset \mathbb{R}_+^{n_q}$. Let $\tilde{\boldsymbol{q}} \in \tilde{\boldsymbol{\mathcal{Q}}} \subset \mathbb{R}_+^{n_{\tilde{q}}}$ be the vector of all other goods that enter the consumer's utility function, with associated nominal market prices $\tilde{\boldsymbol{P}} \in \tilde{\boldsymbol{\mathcal{P}}} \subset \mathbb{R}_+^{n_{\tilde{q}}}$. We continue to let $M \in \mathbb{R}_{++}$ be total expenditure on all goods (income), and define the nominal expenditure on all other goods by $\tilde{M} = \tilde{\boldsymbol{P}}^\mathsf{T}\tilde{\boldsymbol{q}} = M - \boldsymbol{P}^\mathsf{T}\boldsymbol{q} > 0$. For the remainder of the paper, we assume that $n = n_q + n_{\tilde{q}} \geq n_q + 1$, and that expenditure on other goods is strictly positive.

We extend the definition of the nominal expenditure function to

$$E(\boldsymbol{P}, \tilde{\boldsymbol{P}}, u) \equiv \min\left\{\boldsymbol{P}^\mathsf{T}\boldsymbol{q} + \tilde{\boldsymbol{P}}^\mathsf{T}\tilde{\boldsymbol{q}} : u(\boldsymbol{q}, \tilde{\boldsymbol{q}}) \geq u\right\}. \tag{49}$$

We assume that $E : \boldsymbol{\mathcal{P}} \times \tilde{\boldsymbol{\mathcal{P}}} \times \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{M}} \subset \mathbb{R}_+$ is analytic and has neoclassical properties in all prices and the utility index. In particular, it is increasing, 1° homogeneous, and concave in all prices $(\boldsymbol{P}, \tilde{\boldsymbol{P}})$, and increasing in *u*. Denote the Hicksian compensated demands for the goods $\boldsymbol{q}$ by

$$\boldsymbol{g}(\boldsymbol{P}, \tilde{\boldsymbol{P}}, u) = \partial E(\boldsymbol{P}, \tilde{\boldsymbol{P}}, u) / \partial \boldsymbol{P}. \tag{50}$$

We also extend our definition of the indirect utility function to

$$v(\boldsymbol{P}, \tilde{\boldsymbol{P}}, M) \equiv \max\left\{u(\boldsymbol{q}, \tilde{\boldsymbol{q}}) : \boldsymbol{P}^\mathsf{T}\boldsymbol{q} + \tilde{\boldsymbol{P}}^\mathsf{T}\tilde{\boldsymbol{q}} \leq M\right\}, \tag{51}$$

and denote the Marshallian ordinary demands for the goods $\boldsymbol{q}$ by

$$\boldsymbol{h}(\boldsymbol{P}, \tilde{\boldsymbol{P}}, M) = -\frac{\partial v(\boldsymbol{P}, \tilde{\boldsymbol{P}}, M) / \partial \boldsymbol{P}}{\partial v(\boldsymbol{P}, \tilde{\boldsymbol{P}}, M) / \partial M}. \tag{52}$$

The same relationships between the Hicksian and Marshallian demands apply for an incomplete system as for a complete system,

$$\boldsymbol{g}(\boldsymbol{P}, \tilde{\boldsymbol{P}}, u) = \boldsymbol{h}\left(\boldsymbol{P}, \tilde{\boldsymbol{P}}, E(\boldsymbol{P}, \tilde{\boldsymbol{P}}, u)\right), \tag{53}$$

and

$$\boldsymbol{P}^\mathsf{T}\boldsymbol{h}\left(\boldsymbol{P}, \tilde{\boldsymbol{P}}, E(\boldsymbol{P}, \tilde{\boldsymbol{P}}, u)\right) + \tilde{M}\left(\boldsymbol{P}, \tilde{\boldsymbol{P}}, E(\boldsymbol{P}, \tilde{\boldsymbol{P}}, u)\right) = E(\boldsymbol{P}, \tilde{\boldsymbol{P}}, u). \tag{54}$$

LaFrance and Hanemann (1989) demonstrate that an exhaustive list of the properties implied by utility maximization for the subset of Marshallian demands for $\boldsymbol{q}$ are:

(a) they are 0° homogeneous in all prices and income;

(b) they are positive valued;

(c) income strictly exceeds expenditure on $\boldsymbol{q}$; and

(d) the $n_q \times n_q$ matrix of Slutsky substitution terms, $\partial \boldsymbol{h} / \partial \boldsymbol{P}^{\top} + (\partial \boldsymbol{h} / \partial M) \boldsymbol{h}^{\top}$, is symmetric and negative semidefinite.

They also show that (a)–(d) are equivalent to:

(i) the existence of a *quasi-expenditure function*,

$$\hat{E}\left(\boldsymbol{P}, \tilde{\boldsymbol{P}}, \theta(\tilde{\boldsymbol{P}}, u)\right) = E(\boldsymbol{P}, \tilde{\boldsymbol{P}}, u),$$

which is increasing in ($\boldsymbol{P}$,θ), concave in $\boldsymbol{P}$, and satisfies Hotelling's lemma;

(ii) the existence of a *quasi-indirect utility function*,

$$\theta(\tilde{\boldsymbol{P}}, u) = \upsilon(\boldsymbol{P}, \tilde{\boldsymbol{P}}, M),$$

where υ is the inverse of $\hat{E}$ with respect to θ, which is decreasing and quasiconvex in $\boldsymbol{P}$, increasing in M, satisfies Roy's identity, and is related to the indirect utility function by

$$v(\boldsymbol{P}, \tilde{\boldsymbol{P}}, M) = \psi\left(\upsilon(\boldsymbol{P}, \tilde{\boldsymbol{P}}, M), \tilde{\boldsymbol{P}}\right)$$

where ψ is the inverse of θ with respect to υ; and

(iii) the existence of a *quasi-utility function*,

$$\omega(\boldsymbol{q}, \tilde{M}, \tilde{\boldsymbol{P}}) = \min_{\boldsymbol{P}, M} \left\{ \upsilon(\boldsymbol{P}, \tilde{\boldsymbol{P}}, M) : \boldsymbol{P}^{\top}\boldsymbol{q} + \tilde{M} = M \right\},$$

which is increasing and quasiconcave in $(\boldsymbol{q}, \tilde{M})$, conveys the conditional preference map for $\boldsymbol{q}$ given $\tilde{\boldsymbol{q}}$, is related to the *variable indirect utility function* (Diewert 1976, Epstein 1975) by

$$\psi\left(\omega(\boldsymbol{q}, \tilde{M}, \tilde{\boldsymbol{P}}), \tilde{\boldsymbol{P}}\right) = \max_{\tilde{\boldsymbol{q}}} \left\{ u(\boldsymbol{q}, \tilde{\boldsymbol{q}}) : \tilde{\boldsymbol{P}}^{\top}\tilde{\boldsymbol{q}} = \tilde{M} \right\},$$

and to the direct utility function by

$$u(\boldsymbol{q}, \tilde{\boldsymbol{q}}) = \min_{\tilde{\boldsymbol{P}}, \tilde{M}} \left\{ \psi\left(\omega(\boldsymbol{q}, \tilde{M}, \tilde{\boldsymbol{P}}), \tilde{\boldsymbol{P}}\right) : \tilde{\boldsymbol{P}}^{\top}\tilde{\boldsymbol{q}} = \tilde{M} \right\}.$$

LaFrance and Hanemann call (a)–(d) *weak integrability* and show that this concept of integrability for incomplete demand systems completely exhausts the implications of utility maximization. In particular, Epstein's (1982) conditions for the *global integrability* of an incomplete demand system are neither implied by utility maximization nor based on a theoretical construct that is part of the theory of consumer choice. Finally, both the quasi-expenditure and quasi-indirect utility functions convey sufficient information about consumer preferences to admit exact welfare measurement of a change in the prices of the goods of primary interest and income. Consequently, a coherently specified incomplete demand model contains all of the necessary information required to complete any of the usual tasks of applied economic analysis.

Moreover, this theory is completely general; nothing additional has to be assumed about the functional structure of the underlying utility function, $u(\boldsymbol{q},\tilde{\boldsymbol{q}})$, or the indirect preference functions $E(\boldsymbol{P},\tilde{\boldsymbol{P}},u)$, or $v(\boldsymbol{P},\tilde{\boldsymbol{P}},M)$ beyond the standard conditions for utility maximization subject to a linear budget constraint. In effect, the incomplete demand system is augmented by a numeraire composite commodity, $\tilde{M}$, to obtain the budget condition, and we act as if this constitutes a complete system without any loss in generality. The demand for the "good" $\tilde{M}$ may or may not have the same functional form as the demands for the goods of interest, $\boldsymbol{q}$. This provides an additional degree of flexibility that significantly expands the class of theoretically consistent functional forms for the demands of the goods $\boldsymbol{q}$ (LaFrance 1985, 2004; von Haefen 2003).

However, we cannot recover the structure of the demands for $\tilde{\boldsymbol{q}}$, or the structure of that part of consumer's preferences with respect to those goods or their prices, except for any information relating to how those prices influence the demands for $\boldsymbol{q}$. As we will see from the example below, however, this often will not be much cause for concern for the following reason. In many – perhaps most – cases, a careful choice of functional form for the demands of the goods being modeled permits the maximum flexibility with respect to the structure of the demands for the goods that are not modeled.

3.2 The Role of Homogeneity

As generally is the case in a consumer choice problem, the budget set,

$$\mathscr{B}(\boldsymbol{P},\tilde{\boldsymbol{P}},M)\equiv\left\{(\boldsymbol{q},\tilde{\boldsymbol{q}})\in\mathbb{R}_{+}^{n_q}\times\mathbb{R}_{+}^{n_{\tilde{q}}}:\boldsymbol{P}^{\top}\boldsymbol{q}+\tilde{\boldsymbol{P}}^{\top}\tilde{\boldsymbol{q}}\le M\right\}$$

is 0° homogeneous in all prices and income. This implies that we can divide all prices and income by any scalar and the optimal demands for $\boldsymbol{q}$ and $\tilde{\boldsymbol{q}}$ will be left unchanged. Most empirical applications employ some type of general price deflator to reflect the cost of other goods. It is convenient to use normalized prices and income in demand models to impose homogeneity. However, it is important to correctly model the way the prices $\boldsymbol{P}$ affect the demands for the goods $\boldsymbol{q}$ and not to confuse or ignore any influences, no matter how small we might assume they are, that can be masked by a general price deflator. This is particularly important when we are considering the role of symmetry, as in the present case.

As a result of these considerations, it turns out to be very flexible and extremely convenient to normalize by a 1° homogeneous function of the other goods' prices. Therefore, let $\pi : \tilde{\mathcal{P}} \to \mathbb{R}_+$ be a known, non-decreasing (strictly increasing in at least one element of $\tilde{\boldsymbol{P}}$), 1° homogeneous, and concave function $\tilde{\boldsymbol{P}}$. Define normalized prices and income by $\boldsymbol{p} = \boldsymbol{P} \big/ \pi(\tilde{\boldsymbol{P}})$, $\tilde{\boldsymbol{p}} = \tilde{\boldsymbol{P}} \big/ \pi(\tilde{\boldsymbol{P}})$, and $m = M \big/ \pi(\tilde{\boldsymbol{P}})$. Without any loss in generality, then, we can define the *normalized expenditure function* by[10]

$$e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u) \equiv E\left(\boldsymbol{P} \big/ \pi(\tilde{\boldsymbol{P}}), \tilde{\boldsymbol{P}} \big/ \pi(\tilde{\boldsymbol{P}}), u\right) \Big/ \pi(\tilde{\boldsymbol{P}}). \tag{55}$$

It follows that $e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u)$ is increasing in ($\boldsymbol{p}$,u), concave in $\boldsymbol{p}$, is not 1° homogeneous in $\boldsymbol{p}$, the demands for the goods $\boldsymbol{q}$ satisfy Hotelling's lemma,

$$\partial e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u) / \partial \boldsymbol{p} = \boldsymbol{h}\left(\boldsymbol{p}, \tilde{\boldsymbol{p}}, e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u)\right), \tag{56}$$

and the total normalized expenditure on $\boldsymbol{q}$ satisfies the inequality

$$\boldsymbol{p}^{\top} \boldsymbol{h}\left(\boldsymbol{p}, \tilde{\boldsymbol{p}}, e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u)\right) < e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u). \tag{57}$$

This is the form that we choose to model incomplete demand systems for the purpose of extending Gorman systems. The advantage is that homogeneity is automatically accommodated. Since adding up also does not apply, these two forces no longer restrict the functional forms of coherent demand models. We show in the next section that this greatly increases the space of incomplete Gorman systems relative to complete systems. However, as we also will see in that section, the rank of an incomplete Gorman system with prices and income normalized by a function other goods' prices can have a rank that is no greater than three. This might appear to contradict Lewbel's (1987, 1990) result that

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[10] Since $\partial e(\boldsymbol{p}, \tilde{\boldsymbol{p}}, u) \big/ \partial \boldsymbol{p} \equiv \partial E(\boldsymbol{P}, \tilde{\boldsymbol{P}}, u) \big/ \partial \boldsymbol{P}$, normalizing by $\pi(\tilde{\boldsymbol{P}})$ does not change the functional relationships between E and $\boldsymbol{P}$.

Gorman systems that use deflated income can have rank up to four. However, there is no contradiction and Lewbel's rank result is a primary motivation for our choice of deflator. The reason for this difference is explained clearly in Russell and Farris (1998). The logarithmic derivative of the price index deflating income must be the negative of one of the α_k price vectors. Since $\pi(\tilde{\boldsymbol{P}})$ does not involve any of the prices $\boldsymbol{P}$, this property cannot be satisfied for an incomplete system using income normalized by π.

There are two significant advantages to this approach. First, restating this once again, this implies that homogeneity and adding up do not restrict the functional form of the demands for $\boldsymbol{q}$. As we have seen in the previous section, these two properties are responsible for limiting complete Gorman systems to only three possibilities. In contrast, in the next section we show that there is no restriction on the functional form of an incomplete Gorman system. Second, it turns out that using $\pi(\tilde{\boldsymbol{P}})$ as our normalizing factor reserves the maximum flexibility for both the rank and the functional form of the demands for $\tilde{\boldsymbol{q}}$. Since we have no information on the structure of the demands for $\tilde{\boldsymbol{q}}$ when we only include the demand equations for $\boldsymbol{q}$ in an incomplete system, a minimal set of restrictive prior assumptions on the former is highly desirable. Thus, for the goods $\boldsymbol{q}$ we trade off at most one degree of rank (from a maximum of four to a maximum of three) in exchange for complete flexibility in the functional form, while for the goods $\tilde{\boldsymbol{q}}$ we reserve the potential to achieve the maximum possible rank as well as the greatest flexibility for the functional form. These both are illustrated clearly by the following example.

3.3 A Motivating Example

The example we choose to clarify the issues at hand and motivate an interest in incomplete Gorman systems is a special case of the generalized AIDS model recently developed in LaFrance (2004). Define a smooth and increasing function of normalized income by $y = f(m) = m + ½\kappa m^2$, $\kappa > 0$ with a smooth inverse $m(y) = \left(\sqrt{1+2\kappa y} - 1\right)\big/\kappa\ \forall\ m > 0$. Assume that the normalized expenditure function satisfies

$$e(\boldsymbol{p},\tilde{\boldsymbol{p}},u) = m\left(\alpha_0(\tilde{\boldsymbol{p}}) + \alpha(\tilde{\boldsymbol{p}})^\top \boldsymbol{p} + ½\boldsymbol{p}^\top \boldsymbol{B}\boldsymbol{p} - \left(\delta^\top \boldsymbol{p} + \theta(\tilde{\boldsymbol{p}},u)e^{-\gamma^\top \boldsymbol{p}}\right)^{-1}\right). \tag{58}$$

Then Shephard's/Hotelling's Lemma implies that the demands for $\boldsymbol{q}$ are

$$\boldsymbol{q} = \left(\alpha(\tilde{\boldsymbol{p}}) + \boldsymbol{B}\boldsymbol{p}\right)\left(\frac{1}{1+\kappa m}\right) + \gamma\frac{\left[m + ½\kappa m^2 - \left(\alpha_0(\tilde{\boldsymbol{p}}) + \alpha(\tilde{\boldsymbol{p}})^\top \boldsymbol{p} + ½\boldsymbol{p}^\top \boldsymbol{B}\boldsymbol{p}\right)\right]}{(1+\kappa m)}$$

$$+\ (\boldsymbol{I}+\gamma\boldsymbol{p}^{\top})\delta\frac{\left[m+½\kappa m^{2}-\left(\alpha_{0}(\tilde{\boldsymbol{p}})+\alpha(\tilde{\boldsymbol{p}})^{\top}\boldsymbol{p}+½\boldsymbol{p}^{\top}\boldsymbol{B}\boldsymbol{p}\right)\right]^{2}}{(1+\kappa m)}. \tag{59}$$

This is a full rank three subsystem of demands that can be written in Gorman's functionally separable structure with the income functions

$$\left\{1/(1+\kappa m)\quad (m+½\kappa m^{2})/(1+\kappa m)\quad (m+½\kappa m^{2})^{2}/(1+\kappa m)\right\}.$$

None have the form m, $m^{1\pm\kappa}$, $m(\ln m)^{k}$, or $m^{1\pm\iota\tau}$ required in a complete Gorman system, nor can they be reduced to this for any κ > 0, clearly illustrating the functional form argument. Indeed, we could choose *any* sufficiently smooth and strictly increasing function $y=f(m)$ and apply it to (58) to obtain a comparable expression for (59) with exactly three income terms of the form $\left\{1/f'(m)\quad f(m)/f'(m)\quad f(m)^{2}/f'(m)\right\}$. All of these will be legitimate, full rank three, Gorman subsystems.

In addition, the normalized expenditure on the other goods is

$$\begin{aligned}\tilde{m} &= m-\left[\alpha(\tilde{\boldsymbol{p}})^{\top}\boldsymbol{p}+\boldsymbol{p}^{\top}\boldsymbol{B}\boldsymbol{p}\right]\left(\frac{1}{1+\delta m}\right)\\ &-\gamma^{\top}\boldsymbol{p}\frac{\left[m+½\delta m^{2}-\left(\alpha_{0}(\tilde{\boldsymbol{p}})+\alpha(\tilde{\boldsymbol{p}})^{\top}\boldsymbol{p}+½\boldsymbol{p}^{\top}\boldsymbol{B}\boldsymbol{p}\right)\right]}{(1+\delta m)}\\ &-(1+\gamma^{\top}\boldsymbol{p})\delta^{\top}\boldsymbol{p}\frac{\left[m+½\delta m^{2}-\left(\alpha_{0}(\tilde{\boldsymbol{p}})+\alpha(\tilde{\boldsymbol{p}})^{\top}\boldsymbol{p}+½\boldsymbol{p}^{\top}\boldsymbol{B}\boldsymbol{p}\right)\right]^{2}}{(1+\delta m)}.\end{aligned} \tag{60}$$

This also does not have the same form as the demands for $\boldsymbol{q}$. In fact, it is a full rank four Engel curve and only m belongs to Gorman's class of functional forms. It is a simple exercise to show that conditions for (59) that make the $n_q\times n_q$ Slutsky matrix for $\boldsymbol{q}$ negative semidefinite are sufficient for the demand system for $(\boldsymbol{q},\tilde{m})$ to be globally regular. If $n_{\tilde{q}}=1$, so that adding up *defines* the last demand function, then omitting that good from an incomplete system of demand equations (i.e., not forcing this good to necessarily have the same functional form as the goods that are being modeled) relaxes the functional form restrictions on all demand equations and reserves full rank four for the demand equation that is not formally modeled, yet can be rationalized quite easily as a globally integrable system as defined by Epstein (1982).

Finally, if $n_{\tilde{q}}>1$, although the individual demands for the goods $\tilde{\boldsymbol{q}}$ cannot be identified from the expenditure equation for these goods, it is nevertheless a simple exercise to identify a set of sufficient conditions that globally rationalize the demands in (59). To see this, suppose that $\tilde{\boldsymbol{q}}$ is separable from $\boldsymbol{q}$ with a conditional indirect utility function that is a member of the class of rank four normalized systems derived by Lewbel (1990). In particular, let

$$\tilde{v}(\tilde{\boldsymbol{P}},\tilde{M}) = \beta_1(\tilde{\boldsymbol{P}}) - \beta_2(\tilde{\boldsymbol{P}}) \Big/ \left[\left(\tilde{M} / \beta_4(\tilde{\boldsymbol{P}}) \right) - \beta_3(\tilde{\boldsymbol{P}}) \right]^2$$

$$= \beta_1(\tilde{\boldsymbol{p}}) - \beta_2(\tilde{\boldsymbol{p}}) \Big/ \left[\left(\tilde{m} / \beta_4(\tilde{\boldsymbol{p}}) \right) - \beta_3(\tilde{\boldsymbol{p}}) \right]^2 = \tilde{v}(\tilde{\boldsymbol{p}},\tilde{m}), \qquad (61)$$

where the second line follows by the 0° homogeneity of (β_1, β_2, β_3) and the 1° homogeneity of β_4. Roy's identity then implies that the conditional demands for $\tilde{\boldsymbol{q}}$ are

$$\tilde{\boldsymbol{q}} = \tilde{\boldsymbol{\alpha}}_0(\tilde{\boldsymbol{p}}) + \tilde{\boldsymbol{\alpha}}_1(\tilde{\boldsymbol{p}})\tilde{m} + \tilde{\boldsymbol{\alpha}}_2(\tilde{\boldsymbol{p}})\tilde{m}^2 + \tilde{\boldsymbol{\alpha}}_3(\tilde{\boldsymbol{p}})\tilde{m}^3, \qquad (62)$$

with appropriate definitions of the price vectors, $\boldsymbol{\alpha}_k(\tilde{\boldsymbol{p}}),\ k = 0,1,2,3$. Note, in particular, that our choice of the deflator π may or may not be the same as the price index β_4, *with no effect* on the functional form in (62). Substituting the solution for $\tilde{m}$ from (60) into the conditional demands for $\tilde{\boldsymbol{q}}$, produces the unconditional demands (e.g., Gorman 1970; or Blackorby, Primont, and Russell 1978). This subsystem of demands will be rank four and have Gorman's functionally separable structure between income and all prices. Every own- and cross-product term of the third order polynomial defined over the functions,

$$\left\{ m \quad 1/(1+\kappa m) \quad (m+½\kappa m^2)\big/(1+\kappa m) \quad (m+½\kappa m^2)^2\big/(1+\kappa m) \right\},$$

plus the constant function – for a total of nineteen different income functions – appears in the demands for $\tilde{\boldsymbol{q}}$. This is clearly very flexible with respect to this group of demands.

But we do not know if the goods $\tilde{\boldsymbol{q}}$ are separable from the goods $\boldsymbol{q}$, much less whether their conditional demands arise from (61). In other words, even this very flexible set of sufficient conditions for the global regularity of a complete system of demands to rationalize (59) could be too restrictive. We simply have no way to know. So we choose not to impose these conditions or any other set of ad hoc assumptions on the demands that are not part of the formal model. However, the point is that by normalizing on $\pi(\tilde{\boldsymbol{P}})$ we reserve the maximum possible flexibility for that part of the consumer choice problem that we do not observe, model, or measure. The next section formalizes these arguments in the most general context possible for incomplete Gorman systems.

4. Incomplete Gorman Systems

Throughout the rest of this paper, we define $\boldsymbol{x}$ to be a vector-valued function of the normalized prices, $\boldsymbol{p}$, $\boldsymbol{x}=\boldsymbol{g}(\boldsymbol{p})$, $g_i \in \mathcal{C}^\infty\ \forall\ i$, $\left|\partial \boldsymbol{g}(\boldsymbol{p})^\intercal / \partial \boldsymbol{p}\right| \neq 0\ \forall\ \boldsymbol{p} \in \mathcal{P} \subset \mathbb{R}_+^{n_q}$, whose inverse is $\boldsymbol{p}(\boldsymbol{x})$. We also redefine y to be a function of normalized income, $y = f(m)$, $f \in \mathcal{C}^\infty$, $f' > 0$, and denote the inverse of f by $m(y)$. We define an *incomplete Gorman system* by the following natural extension of the complete system case:

$$\frac{\partial y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u)}{\partial \boldsymbol{x}} = \sum_{k=1}^{K} \alpha_k(\boldsymbol{x}, \tilde{\boldsymbol{p}}) h_k\left(y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u)\right). \tag{63}$$

4.1 The Main Result

We are now in a position to state our main result for this class of incomplete demand systems. We focus on two important cases. The first part characterizes every full rank case. The second part extends this to reduced rank cases that have the minimal level of degeneracy defined in Russell and Farris (1998).[11]

Proposition 1. *Every full rank weakly integrable incomplete Gorman system has* $K \leq 3$ *and a definition for* $y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) \equiv f\left(e(\boldsymbol{p}(\boldsymbol{x}), \tilde{\boldsymbol{p}}, u)\right)$ *exists such that*

$$\frac{\partial y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u)}{\partial \boldsymbol{x}} = \begin{cases} \alpha_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) & K=1, \\ \alpha_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \alpha_2(\boldsymbol{x}, \tilde{\boldsymbol{p}}) y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) & K=2, \\ \alpha_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \alpha_2(\boldsymbol{x}, \tilde{\boldsymbol{p}}) y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) + \alpha_3(\boldsymbol{x}, \tilde{\boldsymbol{p}}) y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u)^2 & K=3. \end{cases}$$

Conversely, if $K \geq 3$ *and a maximum number of the Jacoby brackets,*

$$h_k(y)h_\ell'(y) - h_k'(y)h_\ell(y),\ k < \ell,$$

can be written as linear combinations of the $\{h_1(y) \cdots h_K(y)\}$,

$$h_k(y)h_\ell'(y) - h_k'(y)h_\ell(y) = d_{k\ell}^1 h_1(y) + \cdots d_{k\ell}^K h_K(y),\ k < \ell,$$

where the $d_{k\ell}^j$ *are constant, then* $\mathit{rank}\left[\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}})\right] = 3$, *and a definition for y exists*

[11] It also can be shown, using the methods in the Appendix, that the rank of (63) in general can be no greater than three and that this property follows purely from symmetry.

such that

$$\frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)}{\partial \boldsymbol{x}} = \alpha_1(\boldsymbol{x},\tilde{\boldsymbol{p}}) + \alpha_2(\boldsymbol{x},\tilde{\boldsymbol{p}})y(\boldsymbol{x},\tilde{\boldsymbol{p}},u) + \cdots + \alpha_K(\boldsymbol{x},\tilde{\boldsymbol{p}})y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)^{K-1}.$$

We offer a detailed proof of this result in the Appendix, and only briefly outline the main steps here. The proof hinges on the invariance property of Lemma 1 and a small number of results taken from the differential geometry of this class of partial differential equations. First, we exploit the monotonicity of the expenditure function to conclude that at least one of the income functions must satisfy $h'_k(y) \neq 0$. Without loss in generality, let this be $h_1(y)$. We then define the variable $\gamma(y) \equiv \int^y ds/h_1(s)$, so that $\gamma'(y) \equiv 1/h_1(y)$ by the fundamental theorem of calculus.[12] We can now rewrite (63) as

$$\begin{aligned}\frac{\partial \gamma(y(\boldsymbol{x},\tilde{\boldsymbol{p}},u))}{\partial \boldsymbol{x}} &= \gamma'(y(\boldsymbol{x},\tilde{\boldsymbol{p}},u))\frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)}{\partial \boldsymbol{x}} \\ = \frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)/\partial \boldsymbol{x}}{h'_1\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)\right)} &= \alpha_1(\boldsymbol{x},\tilde{\boldsymbol{p}}) + \sum_{k=2}^{K}\alpha_k(\boldsymbol{x},\tilde{\boldsymbol{p}})\frac{h_k\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)\right)}{h_1\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)\right)} \\ &\equiv \alpha_1(\boldsymbol{x},\tilde{\boldsymbol{p}}) + \sum_{k=2}^{K}\alpha_k(\boldsymbol{x},\tilde{\boldsymbol{p}})\tilde{h}_k\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)\right).\end{aligned} \tag{64}$$

Therefore, redefining y by the composite function $\gamma(f(m))$ does not affect integrability or the multiplicative separability of $\boldsymbol{p}$ and y. This lets us focus on the integrability of[13]

$$\frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)}{\partial \boldsymbol{x}} = \alpha_1(\boldsymbol{x},\tilde{\boldsymbol{p}}) + \sum_{k=2}^{K}\alpha_k(\boldsymbol{x},\tilde{\boldsymbol{p}})h_k\left(y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)\right). \tag{65}$$

Invoking symmetry and some simple algebra lets us write a system of ½$n_q(n_q–1)$ equations in the ½$K(K–1)$ Jacoby brackets, $h'_k(y)h_\ell(y) - h_k(y)h'_\ell(y)$,

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[12] One difference between the approach followed here, using the methods of Lie (Hermann 1975), and the one followed by Gorman (1981) is the following. Gorman chose the coordinate system to be the natural logarithms of prices and income, with budget shares on the left-hand-side. In many ways, this is natural for a complete system because adding up then implies that one of the income functions must be constant. In the present case, we can not use adding up to obtain this property. Instead, we undertake a change of variables to redefine y in such a way that, without loss in generality, one of the income functions is constant.

[13] To minimize the notation carried along in this discussion, we often omit tildes and redefine y and other functions without relabeling when it does not cause confusion.

$$\sum_{k=2}^{K}\sum_{\ell=1}^{k-1}\left(\alpha_{ik}\alpha_{j\ell}-\alpha_{jk}\alpha_{i\ell}\right)\left(h_k' h_\ell - h_k h_\ell'\right)=\sum_{k=1}^{K}\left(\frac{\partial\alpha_{jk}}{\partial x_i}-\frac{\partial\alpha_{ik}}{\partial x_j}\right)h_k,\ \forall\ 1\le j<i=2,\cdots,n_q.^{14} \quad (66)$$

The nature of the symmetry conditions is easier to see in matrix form, $\boldsymbol{B}\tilde{\boldsymbol{h}}=\boldsymbol{C}\boldsymbol{h}$, where $\boldsymbol{B}$ is $½n_q(n_q-1)\times½K(K-1)$, $\boldsymbol{C}$ is $½n_q(n_q-1)\times K$, $\boldsymbol{h}$ is $K\times 1$, and $\tilde{\boldsymbol{h}}$ is $½K(K-1)\times 1$. For this to be a well-posed system, therefore, we must have at least as many equations as unknowns, i.e., $n_q \ge K$, and we assume this is so throughout. Premultiplying both sides by $\boldsymbol{B}^\top$ then generates an equivalent square system, $\boldsymbol{B}^\top\boldsymbol{B}\tilde{\boldsymbol{h}}=\boldsymbol{B}^\top\boldsymbol{C}\boldsymbol{h}$. This reveals the crux of the rank condition. $\boldsymbol{B}^\top\boldsymbol{B}$ inherits its rank from $\boldsymbol{A}$, which is K in the full rank case, and has dimension $½K(K-1)\times½K(K-1)$. The existence of a unique solution for $\tilde{\boldsymbol{h}}$ in terms of $\boldsymbol{h}$ therefore requires $K\le 3$.

When $\boldsymbol{B}^\top\boldsymbol{B}$ has full rank, the well-known least squares formula gives $\tilde{\boldsymbol{h}}$ uniquely in terms of $\boldsymbol{h}$ as $\tilde{\boldsymbol{h}}=(\boldsymbol{B}^\top\boldsymbol{B})^{-1}\boldsymbol{B}^\top\boldsymbol{C}\boldsymbol{h}\equiv\boldsymbol{D}\boldsymbol{h}$. We note that $\tilde{\boldsymbol{h}}$ and $\boldsymbol{h}$ depend only on y, but not on $(\boldsymbol{x},\tilde{\boldsymbol{p}})$, while $\boldsymbol{D}$ depends only on $(\boldsymbol{x},\tilde{\boldsymbol{p}})$, but not on y. This implies that the elements of $\boldsymbol{D}$ are constant – they do not depend on $\boldsymbol{x}$, $\tilde{\boldsymbol{p}}$, or y in any way. Linear independence of the income functions and existence of a unique solution also imply that not all of the elements in any row of $\boldsymbol{D}$ can vanish.

Also note that $\boldsymbol{D}$ has dimension $½K(K-1)\times K$. When $K = 1$, $\boldsymbol{D}$ has zero rows and there are no Jacoby brackets. When $K = 2$, $\boldsymbol{D}$ has one row and two columns. When $K = 3$, $\boldsymbol{D}$ has three rows and three columns. If $K > 3$, $\boldsymbol{D}$ has more rows than columns, so that there are more Jacoby brackets than income functions. The simple least squares formula cannot be applied to find $\tilde{\boldsymbol{h}}$ in terms of $\boldsymbol{h}$ since $\boldsymbol{B}^\top\boldsymbol{B}$ must then be singular. The main question in this case is whether there are any redundant equations in the under–identified system $\boldsymbol{B}^\top\boldsymbol{B}\tilde{\boldsymbol{h}}=\boldsymbol{B}^\top\boldsymbol{C}\boldsymbol{h}$, and if so how many. We will return to this momentarily.

The next step is to identify the representation for each full rank case. This is accomplished by combining (65) with the solutions for the Jacoby brackets $\tilde{\boldsymbol{h}}$ to obtain a complete system of linear, first-order, ordinary differential equations with constant coefficients subject to a set of linear side conditions. Rank one follows immediately from (65) with $K = 1$. Rank two is only a little more involved. Starting with the equation,

$$h_1(y)h_2'(y)-h_1'(y)h_2(y)=d_{12}^1 h_1(y)+d_{12}^2 h_2(y),$$

---

[14] In contrast, Gorman (1981) constructed a basis for the vector space spanned by the elements of $\boldsymbol{h}$ and as many product terms $h_k'(y)h_\ell(y)$ as necessary to be able to express each product term $h_k'(y)h_\ell(y)$ as a linear combination of that basis. This is a second difference between the approaches of Lie and Gorman.

note that without loss in generality, we can let $d_{12}^1 \neq 0$. We make a change of variables to

$$\tilde{h}_1(y) = d_{12}^1 h_1(y) + d_{12}^2 h_2(y) \;\Rightarrow\; \tilde{h}_1'(y) = d_{12}^1 h_1'(y) + d_{12}^2 h_2'(y),$$

and

$$\tilde{h}_2(y) = h_2(y) \big/ d_{12}^1 \;\Rightarrow\; \tilde{h}_2'(y) = h_2'(y) \big/ d_{12}^1 .$$

This merely redefines the price vectors with a pair of offsetting linear transformations, $\tilde{\alpha}_1 = \alpha_1 \big/ d_{12}^1$, and $\tilde{\alpha}_2 = -(d_{12}^2 \big/ d_{12}^1)\alpha_1 + d_{12}^1 \alpha_2$. From linear algebra we know this does not affect the rank of ***A***. We also know from Lemma 1 that this does not affect integrability. Some straightforward algebra then implies

$$\tilde{h}_1(y)\tilde{h}_2'(y) - \tilde{h}_1'(y)\tilde{h}_2(y) = \tilde{h}_1(y),$$

and direct integration gives

$$\tilde{h}_2(y) = \tilde{h}_1(y) \int^y ds \big/ \tilde{h}_1(s) .$$

Dropping the tildes and applying the above definition of $\gamma(y)$ produces the representation in the proposition for $K = 2$.

Rank 3 is again more complicated than rank 2, and we leave many details to the Appendix. However, the main ideas are the same. We make a of change variables with the composite function $\gamma(f(m))$ to imply that a representation for y exists that includes the constant function as one of the income terms. We then obtain a system of three equations in three unknowns for the three unique and nontrivial Jacoby brackets in the form

$$h_2'(y) = d_{12}^1 + d_{12}^2 h_2(y) + d_{12}^3 h_3(y),$$

$$h_3'(y) = d_{13}^1 + d_{13}^2 h_2(y) + d_{13}^3 h_3(y), \tag{67}$$

$$h_2(y)h_3'(y) - h_2'(y)h_3(y) = d_{23}^1 + d_{23}^2 h_2(y) + d_{23}^3 h_3(y).$$

The first pair of equations constitutes a complete system of linear, ordinary, first-order differential equations with constant coefficients. These are straightforward to solve for any ***D***. The third equation is a constraint on the set of matrices ***D*** that are compatible with integrability. Solving the first two equations and checking the third for consistency, the only possible case is repeated vanishing roots. The complete solution then takes the form

$$h_k(y) = a_k + b_k y + c_k y^2,\; k = 2,3, \tag{68}$$

for constants $\{a_k, b_k, c_k\}_{k=2}^3$. Redefining the price functions by $\tilde{\alpha}_1 = \alpha_1 + a_2\alpha_2 + a_3\alpha_3$,

$\tilde{\alpha}_2 = b_2\alpha_2 + b_3\alpha_3$, and $\tilde{\alpha}_3 = c_2\alpha_2 + c_3\alpha_3$ then gives the representation for $K = 3$. This completes the proof of the full rank case.

This part of the proposition states that a definition for $y = f(m)$ can always be found such that every full rank weakly integrable Gorman system is at most a quadratic form. This property holds whether or not the system is complete.

The proof for the reduced rank case when $K \geq 3$ has two parts. The first part is to show that we can find a polynomial representation for the demand equations under the stated conditions. This part closely follows Russell and Farris (1998: 193-94). Beginning with a full rank three system, we add a fourth income function. At most two of the three new Jacoby brackets can be in the vector space spanned by $\{1 \ y \ y^2 \ h_4(y)\}$. Two of them are in this space if and only if $h_4(y) = y^3$. Otherwise symmetry is contradicted. The arguments are much the same as for the full rank case, although fewer steps are needed and they are simpler. In particular, we add a linear, first-order, ordinary differential equation with constant coefficients of the form

$$h_4'(y) = d_{14}^1 + d_{14}^2 y + d_{14}^3 y^2 + d_{14}^4 h_4(y), \tag{69}$$

and with no loss in generality, one side condition due to symmetry of the form

$$yh_4'(y) - h_4(y) = d_{24}^1 + d_{24}^2 y + d_{24}^3 y^2 + d_{24}^4 h_4(y). \tag{70}$$

Both of these are compatible if and only if $d_{14}^4 = 0$. This then implies, again with no loss in generality (by constructing linear combinations of the α_k's similar to those constructed above), that $h_4(y) = y^3$. A simple induction then completes this part of the proof.

This part of the proposition states that polynomials produce a maximal number of redundancies and a minimal number of defects for any Gorman system. This property also holds whether or not the system is complete.

Note that this result is based on a very weak condition. As we increase the number of elements in $\boldsymbol{h}$ from K to K+1, it only requires that two of the new Jacoby brackets out of K candidates are contained in the vector space spanned by the K+1 income functions. This is a weak condition because it completely ignores whether any previous Jacoby brackets that are outside of previously defined vector spaces with lower dimensions also are contained in the new K+1 dimensional vector space.

This part of the proof elucidates the issue of the number of *redundancies* and *defects* in the system of equations $\boldsymbol{B}^\top \boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{C}\boldsymbol{h}$ when the left-hand-side matrix is singular. A *redundancy* is a Jacoby bracket that is a linear combination of the elements of $\boldsymbol{h}$ and also can be written as a linear combination of the remaining Jacoby brackets. For example, in the four term case,

$$h_1(y)h_4'(y) - h_1(y)h_4'(y) = 1(3y^2) - 0(y^3) = 3y^2 = h_4'(y),$$

while
$$h_2(y)h_3'(y) - h_2'(y)h_3(y) = y(2y) - 1(y^2) = y^2,$$

and these two equations form one redundancy. In essence, a redundancy repeats another implication of symmetry. A *defect* is a Jacoby bracket that cannot be written as a linear combination of the elements of $\boldsymbol{h}$. Defects do not lie in the linear vector space spanned by the elements of $\boldsymbol{h}$. Again, in the four income term case, note that

$$h_3(y)h_4'(y) - h_3'(y)h_4(y) = y^2(3y^2) - (2y)y^3 = y^4$$

is not contained in the linear vector space spanned by the basis $\{1\ y\ y^2\ \ y^3\}$. Thus, in the case of four income terms, one symmetry condition is redundant and one implies a restriction across the price functions α_k.

In general, the system of equations we have constructed to reflect the symmetry conditions is linear in $\tilde{\boldsymbol{h}}$ and $\boldsymbol{h}$. If symmetry is not contradicted, ony two possibilities exist. If the square, symmetric, positive semidefinite matrix $\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}$ is nonsingular, then each Jacoby bracket can be written as a linear combination of the income functions. This is the full rank case already analyzed. If $\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}$ is singular, then some brackets may be redundant and some may be defects. Redundant brackets add no information. Defects are associated with rows of zeroes in $\boldsymbol{B}$ and $\boldsymbol{C}$, leading to restrictions between the price functions.[15]

We can identify the relationships among the number of Jacoby brackets, spanned brackets, redundancies, and defects in the case of a polynomial representation. A polynomial Jacoby bracket of the form $h_k h_\ell' - h_k' h_\ell = (\ell - k)y^{k+\ell-3}$, $k < \ell$, will be contained in the vector space spanned by $\{1\, y\, y^2 \cdots y^{K-1}\}$ if and only if $k + \ell \le K + 2$. The difference between the number of spanned and redundant brackets is always equal to the number of income terms. The number of income terms, Jacoby brackets, Jacoby brackets spanned

[15] This follows from the representation being unique (i.e., the linear independence of $\boldsymbol{A}$ and $\boldsymbol{h}$ over the set of K–dimensional constants). Obtaining this form for $\boldsymbol{B}$ and $\boldsymbol{C}$ generally will require a series of linear transformations of $\boldsymbol{h}$. These in turn translate into bilinear changes in $\tilde{\boldsymbol{h}}$, offsetting linear changes in $\boldsymbol{A}$, and a series of bilinear changes in $\boldsymbol{B}$ and linear changes in $\boldsymbol{C}$. The full rank two and three cases discussed above illustrate the precise nature of changes in the basis of this type. Any such transformation has no effect on the rank of $\boldsymbol{A}$ or the integrability of the system.

by $\{1\, y\, y^2 \cdots y^{K-1}\}$, the number of *defects* (Jacoby brackets not in this space), and the number of *redundancies* (Jacoby brackets in the space that repeat powers of *y*) is presented in table 1 for all values of $K < \infty$.

Table 1. Number of Income Functions, Jacoby brackets, Defects, and Redundancies.

K	**Jacoby brackets**	**Spanned Brackets**	**Defects**	**Redundancies**
1	0	0	0	0
2	1	1	0	0
3	3	3	0	0
4	6	5	1	1
5	10	8	2	3
6	15	11	4	5
7	21	15	6	8
⋮	⋮	⋮	⋮	⋮
K even	$\frac{1}{2}K(K-1)$	$\frac{1}{4}K(K+2)-1$	$\frac{1}{4}K(K-4)+1$	$\frac{1}{4}K(K-2)-1$
K odd	$\frac{1}{2}K(K-1)$	$\frac{1}{4}(K+1)^2-1$	$\frac{1}{4}(K+1)(K-5)+2$	$\frac{1}{4}(K-1)^2-1$

The last part of the proof is to show that any polynomial system with $K \geq 3$ has $rank[\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}})] \leq 3$. This part of the argument is constructive and only relies on the continuity of the symmetry conditions for powers of *y* from *K*+1 to 2*K*–1. This in turn implies that if the system has a polynomial representation of the form $\partial y/\partial \boldsymbol{x} = \sum_{k=1}^{K} \alpha_k y^{k-1}$, then the matrix of price vectors must satisfy the conditions $\alpha_k \equiv \varphi_k \alpha_K$, $\forall\ k \geq 3$, for some functions $\varphi_k : \mathscr{P} \times \tilde{\mathscr{P}} \to \mathbb{R}$, $\forall k \geq 3$. ■

4.2 Indirect Preferences

Given Proposition 1 characterizing all full rank Gorman systems, we can identify closed form solutions for all of the indirect preferences in the full rank cases. For this purpose, it is sufficient to recover the transformed, normalized expenditure function as (again, a complete set of detailed derivations are contained in the Appendix),

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)=\begin{cases}\beta_1(\boldsymbol{x},\tilde{\boldsymbol{p}})+\theta(\tilde{\boldsymbol{p}},u) & K=1,\\ \beta_1(\boldsymbol{x},\tilde{\boldsymbol{p}})+\beta_2(\boldsymbol{x},\tilde{\boldsymbol{p}})\theta(\tilde{\boldsymbol{p}},u) & K=2,\\ \beta_1(\boldsymbol{x},\tilde{\boldsymbol{p}})-\dfrac{\beta_2(\boldsymbol{x},\tilde{\boldsymbol{p}})}{\left[\theta(\tilde{\boldsymbol{p}},u)-\beta_3(\boldsymbol{x},\tilde{\boldsymbol{p}})\right]} & K=3,\ \lambda\le 0,\\ \beta_1(\boldsymbol{x},\tilde{\boldsymbol{p}})-\dfrac{\beta_2(\boldsymbol{x},\tilde{\boldsymbol{p}})}{\tan\left(\theta(\tilde{\boldsymbol{p}},u)-\beta_3(\boldsymbol{x},\tilde{\boldsymbol{p}})\right)} & K=3,\ \lambda>0,\end{cases},\tag{71}$$

where $\beta_1,\beta_2,\beta_3:\mathscr{P}\times\tilde{\mathscr{P}}\to\mathbb{R}$, $\theta:\tilde{\mathscr{P}}\times\mathscr{U}\to\mathbb{R}$, and λ is the constant term in the integral $\varphi(z)=\int^z ds/(1+\lambda s^2)$ in Lewbel (1987, 1990) and van Daal and Merkies (1989). The cases $\lambda\le 0$ and $\lambda>0$ represent the real and complex roots cases, respectively, for the *Ricatti system* of partial differential equations,

$$\partial z/\partial\boldsymbol{x}=(1+\lambda z^2)\,\partial\gamma_3/\partial\boldsymbol{x}.\tag{72}$$

Here $z=-\gamma_2/(y-\gamma_1)$ and $\gamma_1,\gamma_2,\gamma_3:\mathscr{P}\times\tilde{\mathscr{P}}\to\mathbb{R}$ are price functions that extend the corresponding price functions in van Daal and Merkies (1989) to the incomplete systems case. When the roots are real, we have $\beta_1=\gamma_1+\kappa\gamma_3$, $\beta_2=2\kappa\gamma_3$, $\beta_3=e^{2\gamma_2}$, with $-\lambda=\kappa^2$. In the complex roots case, we have $\beta_1=\gamma_1$, $\beta_2=\kappa\gamma_2$, and $\beta_3=\kappa\gamma_3$, with $\lambda=-(\iota\kappa)^2$ and $\iota=\sqrt{-1}$ as before.

The above results do not preclude higher order polynomials, only more than three income terms with a matrix of *linearly independent* price functions. We demonstrate this with an example extending the one presented by Jerison (1993). In fact, this turns out to be the case that nests the set of all non-trigonometric full rank cases and all reduced rank cases that are polynomials in *y* within a class of demand models that are analytic in *y*. Let the indirect utility function be

$$v(\boldsymbol{x},\tilde{\boldsymbol{p}},y)=\upsilon\left[\left(\frac{\beta(\boldsymbol{x},\tilde{\boldsymbol{p}})}{\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}})-y}\right)^{\eta}-\delta(\boldsymbol{x},\tilde{\boldsymbol{p}}),\tilde{\boldsymbol{p}}\right],\tag{73}$$

where we assume $\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}})>y$ for monotonicity and let η be any real number in $[1,\infty)$. By Roy's identity, the incomplete demand system for the goods $\boldsymbol{q}$ is

$$\boldsymbol{q}=m'(y)\left(\frac{\partial\boldsymbol{p}(\boldsymbol{x})^{\top}}{\partial\boldsymbol{x}}\right)^{-1}\left[\frac{\partial\gamma}{\partial\boldsymbol{x}}-\frac{\partial\beta}{\partial\boldsymbol{x}}\left(\frac{\gamma-y}{\beta}\right)+\left(\frac{\beta}{\eta}\right)\frac{\partial\delta}{\partial\boldsymbol{x}}\left(\frac{\gamma-y}{\beta}\right)^{\eta+1}\right].\tag{74}$$

For certain values of η, this takes the form of proposition 1 and illustrates the full nature of its implications. First note that there are precisely three linearly independent functions of y on the right-hand-side of (74). When $\eta = 1$, we have a quadratic in y. If η is an integer greater than one, expanding the last term in square brackets with the binomial formula implies that all powers of y from 0 to $\eta+1$ appear on the right. The demand model cannot be reduced to a quadratic for any $\eta > 1$. The first two terms in square brackets involve the powers 0 and 1 in y and the sub-matrix of price vectors on the powers of y from 2 through $\eta+1$ has rank equal to one.

However, η also can be any real value in $[1,\infty)$ and preferences will remain well–behaved with appropriate choices of the functions $\{\beta,\gamma,\delta\}$. In such a case, the last term in square brackets on the right-hand-side of (74) is analytic that has a convergent Taylor series expansion over the set of positive values for $\gamma - y$. Moreover, the vectors of price functions for all of the powers of y greater than one are proportional and the matrix of price functions, even when there is an infinite number of columns, has rank no greater than three.

Thus, if there is a finite number of income terms, we must have a polynomial in y. If the model has full rank, then this polynomial is at most a quadratic. The duality between indirect preferences and the demand equations implies that this is the only functional form for indirect preferences that nests all non-trigonometric full rank cases, reduced rank polynomial cases, and analytic rank three incomplete demand systems. It is clear that a very large set of well–defined demand models exists beyond quadratics, but that each element can be represented as an irreducible polynomial and might even have an infinite number of income terms, and that each of them has a matrix of prices coefficients that has rank no greater than three.[16]

4.3 The Relationship to Local Transformation Groups

In differential geometry, the space of all real projective transformation groups is commonly associated with the *special linear group two*, $\mathfrak{sl}(2)$. This is generally defined by the set of all 2×2 real matrices,

[16] This argument continues to hold for complete systems simply by returning to the definitions of $\boldsymbol{x}$ and y in section two and applying the functional form restrictions on y.

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

that have a unit determinant, $\alpha\delta - \beta\gamma = 1$. The inverse matices,

$$A^{-1} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix},$$

also are members of $\mathfrak{sl}(2)$, as well as the identiy map $\boldsymbol{I}_2$. Any real projective transformation group can be written in the form

$$y = \frac{\alpha\theta + \beta}{\gamma\theta + \delta} \Leftrightarrow \theta = \frac{\delta y - \beta}{-\gamma y + \alpha},\ \forall\ \alpha\delta - \beta\gamma = 1. \tag{75}$$

The set of all 2×2 matrix inverses in $\mathfrak{sl}(2)$ are one-to-one and onto the inverse functions for this group, and $\boldsymbol{I}_2$ defines the identity map in both spaces.

If we specify that $\alpha, \beta, \gamma : \boldsymbol{\mathcal{X}} \times \tilde{\boldsymbol{\mathcal{P}}} \to \mathbb{R}$ and $\theta : \tilde{\boldsymbol{\mathcal{P}}} \times \boldsymbol{\mathcal{U}} \to \mathbb{R}$, then some simple algebra gives

$$\frac{\partial y}{\partial \boldsymbol{x}} = \left(\alpha\frac{\partial\beta}{\partial \boldsymbol{x}} - \beta\frac{\partial\alpha}{\partial \boldsymbol{x}}\right) + \left[\left(\beta\frac{\partial\gamma}{\partial \boldsymbol{x}} - \gamma\frac{\partial\beta}{\partial \boldsymbol{x}}\right) - \left(\alpha\frac{\partial\delta}{\partial \boldsymbol{x}} - \delta\frac{\partial\alpha}{\partial \boldsymbol{x}}\right)\right] y + \left(\gamma\frac{\partial\delta}{\partial \boldsymbol{x}} - \delta\frac{\partial\gamma}{\partial \boldsymbol{x}}\right) y^2. \tag{76}$$

This representation defines a class of indirect utility functions in the form

$$v(\boldsymbol{p}, \tilde{\boldsymbol{p}}, m) = \upsilon\left\{\frac{\delta(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}) f(m) - \beta(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}})}{-\gamma(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}) f(m) + \alpha(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}})}, \tilde{\boldsymbol{p}}\right\}, \tag{77}$$

that generate incomplete Gorman systems, or equivalently, normalized and transformed expenditure functions in the form

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \frac{\alpha(\boldsymbol{x}, \tilde{\boldsymbol{p}})\theta(\tilde{\boldsymbol{p}}, u) + \beta(\boldsymbol{x}, \tilde{\boldsymbol{p}})}{\gamma(\boldsymbol{x}, \tilde{\boldsymbol{p}})\theta(\tilde{\boldsymbol{p}}, u) + \delta(\boldsymbol{x}, \tilde{\boldsymbol{p}})}. \tag{78}$$

From this we can see immediately the connection between the class of all non-trigonometric full rank three Gorman systems and the projective transformation group with real parameters. Note that $\gamma \neq 0$ is required for a full rank three system, so that we can rescale the price functions to obtain

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \frac{\tilde{\alpha}(\boldsymbol{x}, \tilde{\boldsymbol{p}})\theta(\tilde{\boldsymbol{p}}, u) + \tilde{\beta}(\boldsymbol{x}, \tilde{\boldsymbol{p}})}{\theta(\tilde{\boldsymbol{p}}, u) + \tilde{\delta}(\boldsymbol{x}, \tilde{\boldsymbol{p}})} \tag{79}$$

with $\tilde{\alpha} = \alpha/\gamma$, $\tilde{\beta} = \beta/\gamma$, and $\tilde{\delta} = \delta/\gamma$. It is straightforward to convert (79) to the form given in the third line of (71) by adding and subtracting $\tilde{\alpha}\tilde{\delta}$ in the numerator and rearranging terms. Also note that $\alpha = \delta = 1$ and $\gamma = 0$ gives the rank one case, while $\delta = 1$ and $\gamma = 0$ gives the full rank two case. In the Appendix, we also show that the case of complex roots for the full rank three case generates a member of the complex projective transformation group with complex-valued price functions. Thus, the class of all full rank Gorman systems can be derived from a projective transformation group. The converse is also true – the (transformed and normalized) expenditure function and quasi-indirect utility function of every full rank Gorman system is a member of this group. The group property also applies to complete Gorman systems if we define $\theta = u$, impose the necessary functional form restrictions on *y*, and define α, β, γ, and δ to be functions of the nominal prices $\boldsymbol{P}$ rather than of the normalized prices $(\boldsymbol{p}, \tilde{\boldsymbol{p}})$. This follows from the solutions for all full rank three indirect preferences derived at the end of section two above.

Thus, the essential difference between complete and incomplete Gorman systems, given how we define the latter in this paper, is that (79) is valid for any $\theta(\tilde{\boldsymbol{p}}, u)$ that is smooth in $(\tilde{\boldsymbol{p}}, u)$ and monotone in *u* and for any smooth and monotone $y = f(m)$. The function θ is an arbitrary constant of integration obtained by integrating an incomplete demand system to recover the part of the expenditure function that is associated with the prices $\boldsymbol{p}$. In general, without knowledge of θ's structure – which can only be obtained from the demands for $\tilde{\boldsymbol{q}}$ – it can take any form. Moreover, it is known that (79) holds for any $y = f(m)$, so that there also is no functional form restriction on *y* for an incomplete Gorman system.

5. Nesting Rank and Functional Form

In this section, we present a simple method to nest the rank and the functional form of incomplete Gorman systems. From our characterization of preferences obtained in the previous sections, we found that any non-trigonometric full rank Gorman system can be written as a special case of the following generalization of the QES obtained by Howe, Pollak, and Wales (1979),

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \beta_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) - \frac{\beta_2(\boldsymbol{x}, \tilde{\boldsymbol{p}})}{\left[\theta(\tilde{\boldsymbol{p}}, u) - \beta_3(\boldsymbol{x}, \tilde{\boldsymbol{p}})\right]}, \tag{80}$$

with $\beta_1, \beta_2, \beta_3 : \boldsymbol{\mathcal{X}} \times \tilde{\boldsymbol{\mathcal{P}}} \to \mathbb{R}$, $\theta : \tilde{\boldsymbol{\mathcal{P}}} \times \boldsymbol{\mathcal{U}} \to \mathbb{R}, \partial\theta/\partial u > 0$, and $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{p}), \left|\partial \boldsymbol{g}^{\top}/\partial \boldsymbol{p}\right| \neq 0$. Taking partial derivatives of (80) with respect to the normalized prices $\boldsymbol{p}$ gives

$$\frac{\partial y(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}},u)}{\partial \boldsymbol{p}} = \frac{\partial \boldsymbol{g}(\boldsymbol{p})^{\top}}{\partial \boldsymbol{p}} \times$$

$$\left(\frac{\partial \beta_1(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}})}{\partial \boldsymbol{x}} - \frac{\partial \beta_2(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}})/\partial \boldsymbol{x}}{\left[\theta(\tilde{\boldsymbol{p}},u)-\beta_3(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}})\right]} - \frac{\beta_2(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}})\partial \beta_3(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}})/\partial \boldsymbol{x}}{\left[\theta(\tilde{\boldsymbol{p}},u)-\beta_3(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}})\right]^2} \right). \tag{81}$$

This can be written in the equivalent, but more transparent form,

$$f'(m)\boldsymbol{q} = \frac{\partial \boldsymbol{g}(\boldsymbol{p})^{\top}}{\partial \boldsymbol{p}} \left[\frac{\partial \beta_1}{\partial \boldsymbol{x}} + \frac{\partial \beta_2}{\partial \boldsymbol{x}} \left(\frac{f(m)-\beta_2}{\beta_3} \right) - \beta_2 \frac{\partial \beta_3}{\partial \boldsymbol{x}} \left(\frac{f(m)-\beta_2}{\beta_3} \right)^2 \right], \tag{82}$$

where all terms in the square brackets on the right are evaluated at $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{p})$.

In section two, we found that the set of Box-Cox transformations on income could be used to nest the (non-trigonometric) full rank two and three complete systems. A natural extension to the class of incomplete Gorman systems defined in section four is a Box-Cox transformation of normalized income, say,

$$y = (m^{\kappa}-1)/\kappa,\ \kappa \geq 0. \tag{83}$$

An interesting candidate for the definition of $\boldsymbol{x}$ also is an n_q–vector of Box-Cox transformations of the normalized prices,

$$x_i = (p_i^{\lambda}-1)/\lambda,\ i=1,\cdots,n_q,\ \lambda \geq 0. \tag{84}$$

These definitions of $\boldsymbol{x}$ and y are attractive because they permit us to consider a full range of dependent variables from quantities when $\kappa=\lambda=1$ to budget shares when $\kappa=\lambda=0$, since $f'(m)=m^{\kappa-1}$, while $\partial \boldsymbol{g}(\boldsymbol{p})^{\top}/\partial \boldsymbol{p} = \mathbf{diag}\left[p_i^{\lambda-1}\right]$. If we write the incomplete demand system with budget shares as left-hand-side variables,

$$\boldsymbol{w} = m^{-\kappa}\mathbf{diag}\left[p_i^{\lambda}\right] \times \left[\frac{\partial \beta_1}{\partial \boldsymbol{x}} + \frac{\partial \beta_2}{\partial \boldsymbol{x}} \left(\frac{(m^{\kappa}-1)/\kappa-\beta_2}{\beta_3} \right) - \beta_2 \frac{\partial \beta_3}{\partial \boldsymbol{x}} \left(\frac{(m^{\kappa}-1)/\kappa-\beta_2}{\beta_3} \right)^2 \right], \tag{85}$$

where $w_i = p_i q_i/m$ is the budget share of the i^{th} good, and the price functions, β_1, β_2, and β_3 and their derivatives are defined over the above Box-Cox functions of normalized prices. Thus we have a natural procedure to nest the class of Gorman functional forms for all non-trigonometric systems. We also obtain a way to simultaneously nest the functional form of the price variables. Moreover, the rank of the demand model is reflected by the three vectors $\{\partial\beta_1/\partial\boldsymbol{x} \quad \partial\beta_2/\partial\boldsymbol{x} \quad \partial\beta_3/\partial\boldsymbol{x}\}$. Hence, the price functions can be tested for dependence on $\boldsymbol{x}$ to determine the rank of the system. We illustrate this method for two classes of models.

5.1 A Nested Quadratic PIGL/PIGLOG Extension of AIDS

LaFrance (2004) solves the integrablity problem for the linear approximate AIDS model and uses the solution to obtain a nested set of full rank generalized AIDS models. The transformed normalized expenditure function he obtains is

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},u) \equiv \alpha_0(\tilde{\boldsymbol{p}}) + \alpha(\tilde{\boldsymbol{p}})^\top \boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x} - \left(\delta^\top \boldsymbol{x} + \theta(\tilde{\boldsymbol{p}},u)e^{-\gamma^\top \boldsymbol{x}}\right)^{-1}, \tag{86}$$

with the above Box-Cox definitions for $\boldsymbol{x}$ and y. Applying Hotelling's lemma gives the demands in budget share form for the goods $\boldsymbol{q}$ as

$$\boldsymbol{w} = m^{-\kappa}\mathbf{diag}\left[p_i^\lambda\right]\Big\{\alpha(\tilde{\boldsymbol{p}}) + \boldsymbol{B}\boldsymbol{x} + \gamma\left[y - \alpha_0(\tilde{\boldsymbol{p}}) - \alpha(\tilde{\boldsymbol{p}})^\top \boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x}\right]$$

$$+\left(\boldsymbol{I} + \gamma^\top \boldsymbol{x}\right)\delta\left[y - \alpha_0(\tilde{\boldsymbol{p}}) - \alpha(\tilde{\boldsymbol{p}})^\top \boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x}\right]^2\Big\}. \tag{87}$$

As long as α and $\boldsymbol{B}$ do not vanish simultaneously, which is necessary for the model to be able to attain rank three, it follows that: (a) $\gamma \neq \mathbf{0}$ and $\delta \neq \mathbf{0}$ is necessary and sufficient for a full rank three quadratic PIGL/PIGLOG model; (b) $\gamma \neq \mathbf{0}$ and $\delta = \mathbf{0}$ is necessary and sufficient for a full rank two model; (c) $\gamma = \mathbf{0}$ and $\delta \neq 0$ is necessary and sufficient for a rank two system that excludes the linear term in the deflated and transformed superlative income variable; and (d) $\gamma = \delta = \mathbf{0}$ is necessary and sufficient for a rank one homothetic PIGL/PIGLOG system. Thus, we obtain a rich class of models that permits nesting, testing and estimating the rank and functional form of the income terms in incomplete demand systems with a generalized AIDS structure.

5.3 A Nested Quadratic PIGL/PIGLOG Extension of Quadratic Utility

We now apply the same method to produce a PIGL/PIGLOG generalization of quadratic utility. The rank two version of this model in log-log form produces generalized translog indirect preferences (Christensen, Jorgenson, and Lau, 1975). First define the functions

$$\varphi(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x} + 2\gamma^\top \boldsymbol{x} + 1, \tag{88}$$

$$\eta(\boldsymbol{x},\tilde{\boldsymbol{p}}) = \alpha_0(\tilde{\boldsymbol{p}}) - \alpha(\tilde{\boldsymbol{p}})^\top \boldsymbol{x}, \tag{89}$$

where $\alpha(\tilde{\boldsymbol{p}})$ is a vector–valued function of other prices, $\alpha_0(\tilde{\boldsymbol{p}})$ is a real-valued function of other prices, $\boldsymbol{B}$ is a symmetric $n_q \times n_q$ matrix of parameters, and γ is a vector of parameters. The starting point for this application of the nesting procedure is the class of indirect utility functions defined by

$$v(\boldsymbol{x}, \tilde{\boldsymbol{p}}, y) = \upsilon\left\{-\left(\frac{\sqrt{\varphi(\boldsymbol{x}, \tilde{\boldsymbol{p}})}}{y - \eta(\boldsymbol{x}, \tilde{\boldsymbol{p}})}\right) - \left(\frac{\delta^{\mathsf{T}}\boldsymbol{x}}{\sqrt{\varphi(\boldsymbol{x}, \tilde{\boldsymbol{p}})}}\right), \tilde{\boldsymbol{p}}\right\}, \tag{90}$$

which is equivalent to the transformed normalized expenditure function,

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \eta(\boldsymbol{x}, \tilde{\boldsymbol{p}}) - \left(\frac{\varphi(\boldsymbol{x}, \tilde{\boldsymbol{p}})}{\delta^{\mathsf{T}}\boldsymbol{x} + \sqrt{\varphi(\boldsymbol{x}, \tilde{\boldsymbol{p}})}\theta(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u)}\right) \tag{91}$$

Applying the above Box-Cox transformations and Roy's identity gives the incomplete demand equations for the goods $\boldsymbol{q}$ in budget share form as

$$\boldsymbol{w} = m^{-\kappa}\textbf{diag}\left[p_i^{\lambda}\right]\left\{\alpha + \left[1 - \delta^{\mathsf{T}}\boldsymbol{x}\left(\frac{y-\eta}{\varphi}\right)\right]\left(\frac{y-\eta}{\varphi}\right)(\boldsymbol{Bx} + \gamma) + \frac{(y-\eta)^2}{\varphi}\delta\right\}. \tag{92}$$

In this case, $\kappa = \lambda = 0$ gives a rank three extension of a generalized translog type of indirect preferences, $\kappa = \lambda = 1$ gives a generalized quadratic utility type of indirect preferences, and all values of κ and λ produce a PIGL/PIGLOG model that can have rank up to three. Rank two is obtained with $\delta = \boldsymbol{0}$. If $\eta(\boldsymbol{x}, \tilde{\boldsymbol{p}}) \equiv 0$ and $\delta = \boldsymbol{0}$, we have a rank one homothetic model. We again are able to nest both the rank and the functional form of the incomplete system of demand equations within a single unifying framework.

It is useful to view this incomplete demand system in terms of

$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha + \left[1 - \delta^{\mathsf{T}}\boldsymbol{x}\left(\frac{y-\eta}{\varphi}\right)\right]\left(\frac{y-\eta}{\varphi}\right)(\boldsymbol{Bx} + \gamma) + \frac{(y-\eta)^2}{\varphi}\delta\,. \tag{93}$$

The reason is that Lemma 1 above, and Lemmas 4 and 5 in the Appendix, imply that this model lets us determine sufficient parametric restrictions for the global concavity of y in $\boldsymbol{x}$, and hence of e in $\boldsymbol{p}$. Calculating the second-order partial derivatives and careful (and quite tedious) grouping, canceling, and algebraic manipulations give

$$\frac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}^{\mathsf{T}}} = \left[1 - \delta^{\mathsf{T}}\boldsymbol{x}\left(\frac{y-\eta}{\varphi}\right)\right]\left(\frac{y-\eta}{\varphi}\right)\left[\boldsymbol{B} - \left(\frac{1}{\varphi}\right)(\boldsymbol{Bx} + \gamma)(\boldsymbol{Bx} + \gamma)^{\mathsf{T}}\right]$$

$$+\,2\frac{(y-\eta)^3}{\varphi^2}\left[\boldsymbol{I}-\left(\frac{1}{\varphi}\right)(\boldsymbol{Bx}+\gamma)\boldsymbol{x}^{\top}\right]\delta\delta^{\top}\left[\boldsymbol{I}-\left(\frac{1}{\varphi}\right)\boldsymbol{x}(\boldsymbol{Bx}+\gamma)^{\top}\right]. \qquad (94)$$

Symmetry of $\boldsymbol{B}$ is necessary and sufficient for symmetry of $\partial^2 y/\partial\boldsymbol{x}\partial\boldsymbol{x}^{\top}$, and therefore for symmetry of $\partial^2 e/\partial\boldsymbol{p}\partial\boldsymbol{p}^{\top}$. Recall the transformation $\tilde{u}=-u^{-1}$ (from Howe, Pollak, and Wales 1979) of the Gorman polar form of the quasi-indirect utility function when $\delta=\boldsymbol{0}$. That is, we take the negative reciprocal of $(y-\eta)/\sqrt{\varphi}$, which is the generalized quadratic quasi-indirect utility function.[17] In this case $y=\eta$ is the bliss point and monotonicity requires $y-\eta<0$, while $\varphi>0$ is required in Gorman's normalization of preferences. If $\|\delta\|<\varepsilon$ for sufficiently small $\varepsilon > 0$, we also will have $1-\delta^{\top}\boldsymbol{x}(y-\eta)/\varphi>0$. Moreover, this latter inequality must be satisfied in a neighborhood of every point in the interior of the domain of the demand system if preferences are well-behaved. This is equivalent to the condition that if we add $-\delta^{\top}\boldsymbol{x}/\sqrt{\varphi}$ to $\tilde{u}=-u^{-1}$, then we do not change the sign of the utility index. This condition is required for the Howe, Pollak and Wales (1979) transformation from u to $-u^{-1}$ to be well-defined and it is easy to show that preferences will become irregular if it is violated. In any case, we would expect the second-order income effects to be small relative to the first-order income effects. In other words, $\delta^{\top}\boldsymbol{x}$ should be small relative to φ. The upshot is that, if $1-\delta^{\top}\boldsymbol{x}(y-\eta)/\varphi>0$, $y-\eta<0$ and $\varphi>0$, then the second line of (94) is a symmetric, negative semidefinite, rank one matrix. Lemmas 4 and 5 in the Appendix show that $\boldsymbol{B}=\boldsymbol{L}\boldsymbol{L}^{\top}$ is necessary and $\boldsymbol{B}=\boldsymbol{L}\boldsymbol{L}^{\top}+\gamma\gamma^{\top}$ is sufficient, for $\partial^2 y/\partial\boldsymbol{x}\partial\boldsymbol{x}^{\top}$ to be negative semidefinite. Under the conditions of the second half of Lemma 1, this in turn is necessary and sufficient for the weak integrability of this incomplete demand system throughout the open set

$$\Im\equiv\left\{(\boldsymbol{p},\tilde{\boldsymbol{p}},m)\in\mathcal{P}\times\tilde{\mathcal{P}}\times\mathcal{M}:\varphi>0,y-\eta<0,1-\delta^{\top}\boldsymbol{x}(y-\eta)/\varphi>0\right\}. \qquad (95)$$

These restrictions apply only to the model parameters and are not difficult to implement.

6. Conclusions

In this paper, we extend Gorman's theory of aggregation to incomplete demand systems. In contrast to complete demand systems, there is no restriction on the class of functional

[17] We use the name *generalized quadratic* to refer to the fact that indirect preferences are defined in terms of deflated and transformed prices and income, $\boldsymbol{x}$ and y, rather than directly in terms of $\boldsymbol{p}$ and m.

forms for the income variables. On the other hand, the maximal rank of an incomplete Gorman system is three. This follows purely from Slutsky symmetry. We closed a small gap in the literature on complete Gorman systems and in doing so obtained a complete set of closed form solutions for indirect preferences for all full rank and minimally degenerate reduced rank Gorman systems. We also have shown the relationship between Gorman systems of Engel curves and projective transformation groups. Finally, we developed a simple procedure to nest the rank and functional form in any Gorman system. We then used Box-Cox transformations of normalized prices and a Box-Cox transformation of normalized income to generate two large classes of nested functional forms. One nests the rank and functional form of generalized AIDS models. The other applies the same procedure to nest generalized translog and generalized quadratic utility types of indirect preferences.

We have found both frameworks for nesting incomplete demand systems to be empirically tractable as well as substantial improvements over the traditional rank two alternatives (Beatty and LaFrance, 2000; LaFrance, Beatty, Pope and Agnew, 2000, 2002; and LaFrance and Beatty, 2003). In both cases, rank three appears to be essential for many data sets. The point estimates for the Box-Cox parameters on prices and income tend to be closer to unity than zero. Lemma 1 suggests that this may be generic because the logarithmic transformation to define y makes curvature more difficult to implement. However, both restrictions ($\kappa = \lambda = 1$ or 0, respectively) are rejected at the usual levels of significance in the data sets we have used to investigate this question.

Incomplete demand systems have many applications. Permanent income consumption models are easily extended to labor/leisure tradeoffs and nonlinear budget constraints. The demand for consumption goods won't be 0° homogeneous in goods prices and expenditure and won't satisfy adding up. But they will satisfy the properties of an incomplete demand system. This is a natural and flexible way to incorporate labor supply decisions into demand analysis. A second application is the dynamic theory of consumption when individuals face an uncertain life span and have a bequest motive. In this case, the budget constraint is unobservable. But one can use an incomplete demand system to draw inferences on the impact of this on consumption choices. A third application is to production. There is no difference in theory or principle between expenditure and cost functions and everything developed here applies with equal weight to production theory.

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Mathematical Appendix

A.1 Symmetry and Curvature

Lemma 1. *If the expenditure function is twice differentiable on* $\mathcal{P}\times\mathcal{U}\subset\mathbb{R}^n_+\times\mathbb{R}$, $y=f(E),\ f\in\mathcal{C}^2,\ f'>0\ \forall\ M\in\mathcal{M}$, $\boldsymbol{x}=\boldsymbol{g}(\boldsymbol{P})$, $\boldsymbol{g}\in\mathcal{C}^2\ \forall\ \boldsymbol{p}\in\mathcal{P}$, *and the expenditure function satisfies* $E(\boldsymbol{P},u)=M\left(y(\boldsymbol{g}(\boldsymbol{P}),u)\right)$, *then* $\partial^2E(\boldsymbol{P},u)/\partial\boldsymbol{P}\partial\boldsymbol{P}^\top$ *is symmetric at* (***P***,*u*) *if and only if* $\partial^2y(\boldsymbol{g}(\boldsymbol{P}),u)/\partial\boldsymbol{x}\partial\boldsymbol{x}^\top$ *is symmetric at* (***g***(***P***),*u*). *If* $x_i=g_i(P_i), g_i\in\mathcal{C}^2, g_i'>0, g_i''\le 0,\forall\ p_i\in\mathcal{P}_i\subset\mathbb{R}_+,\forall\ i$, $M''\le 0\ \forall\ y\in\mathcal{Y}$ *and y is concave in* ***x***, *then E is concave in* ***P***.

Proof: We have

$$\frac{\partial e}{\partial \boldsymbol{p}}=\phi'\frac{\partial \boldsymbol{g}^\top}{\partial \boldsymbol{p}}\frac{\partial y}{\partial \boldsymbol{x}},\tag{A.1}$$

so that

$$\frac{\partial^2 e}{\partial \boldsymbol{p}\partial \boldsymbol{p}^\top}=\phi''\frac{\partial \boldsymbol{g}^\top}{\partial \boldsymbol{p}}\frac{\partial y}{\partial \boldsymbol{x}}\frac{\partial y}{\partial \boldsymbol{x}^\top}\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{p}^\top}+\phi'\sum_{i=1}^{n}\frac{\partial y}{\partial x_i}\frac{\partial^2 g_i}{\partial \boldsymbol{p}\partial \boldsymbol{p}^\top}+\phi'\frac{\partial \boldsymbol{g}^\top}{\partial \boldsymbol{p}}\frac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}^\top}\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{p}^\top}.\tag{A.2}$$

The first two terms on the right are automatically symmetric, so that symmetry of the left-hand-side is equivalent to symmetry of the Hessian matrix on the far right-hand-side. The first two matrices on the right are negative semidefinite when $\phi''\le 0$ and $g_i''\le 0\ \forall\ i$, so that if $\partial^2 y/\partial\boldsymbol{x}\partial\boldsymbol{x}^\top$ is negative semidefinite, then $\partial^2 e/\partial\boldsymbol{p}\partial\boldsymbol{p}^\top$ is as well. ■

Lemma 2. $f:\mathbb{R}\to\mathbb{R},\ f\in\mathcal{C}^\infty,\ f'>0$, *satisfies the linear first-order ordinary differential equation* $Mf'(M)=a+bf(M)\ \forall\ M\in\mathcal{M}$ *if and only if*

$$f(M)=\begin{cases}a\ln M+c, & \text{if } b=0,\\ cM^b-(a/b), & \text{if } b>0.\end{cases}$$

Proof: Simply integrate the o.d.e. in the two cases. ■

Lemma 3. *If* $z:\mathcal{P}\times\mathcal{U}\to\mathbb{R},\ z\in\mathcal{C}^\infty$ *satisfies the partial differential equations*,

$$\frac{\partial z(\boldsymbol{P},u)}{\partial \boldsymbol{P}}=\left[1+\gamma\left(\beta_2(\boldsymbol{P})\right)z(\boldsymbol{P},u)^2\right]\frac{\partial\beta_2(\boldsymbol{P})}{\partial \boldsymbol{P}},$$

then either $\gamma\left(\beta_2(\boldsymbol{P})\right)\equiv\lambda$ *is constant or*

$$\frac{\partial z(\boldsymbol{P},u)}{\partial \boldsymbol{P}}=\gamma'\left(\beta_2(\boldsymbol{P})\right)\frac{\partial\beta_2(\boldsymbol{P})}{\partial \boldsymbol{P}}\equiv\frac{\partial\gamma\left(\beta_2(\boldsymbol{P})\right)}{\partial \boldsymbol{P}}.$$

Proof: Divide both sides of the system of partial differential equations by the term in square brackets on the right. This implies

$$\frac{\partial z/\partial \boldsymbol{x}}{1+\gamma\left(\beta_2(\boldsymbol{x})\right)z^2}=\frac{\partial\beta_2}{\partial\boldsymbol{x}}.$$

The matrix of second-order partial derivatives on both sides must therefore be symmetric. This in turn implies that

$$\frac{\partial^2 z/\partial\boldsymbol{x}\,\partial\boldsymbol{x}^{\top}}{\left[1+\gamma z^2\right]}-\frac{2\gamma z}{\left[1+\gamma z^2\right]^2}\frac{\partial z}{\partial\boldsymbol{x}}\frac{\partial z}{\partial\boldsymbol{x}^{\top}}-\left(\frac{z}{1+\gamma z^2}\right)^2\gamma'\frac{\partial z}{\partial\boldsymbol{x}}\frac{\partial\beta_2}{\partial\boldsymbol{x}^{\top}}=\frac{\partial^2\beta_2}{\partial\boldsymbol{x}\partial\boldsymbol{x}^{\top}}.$$

The first two matrices on the left and the matrix on the right are automatically symmetric. It follows that $\gamma'(\partial z/\partial\boldsymbol{x})\times(\partial\beta/\partial\boldsymbol{x})^{\top}$ must be symmetric. There are only two ways this can be true: either (1) $\gamma'=0$; or (2) $\partial z/\partial\boldsymbol{x}=\partial\gamma/\partial\boldsymbol{x}$. In the second case we are free to redefine $\gamma:\boldsymbol{\mathcal{X}}\to\mathbb{R}$ by $\gamma(\boldsymbol{x})\equiv\gamma\left(\beta_2(\boldsymbol{x})\right)$. ■

A.2 Differential Equations for PIGL and PIGLOG Functional Forms

Consider the quasi-linear ordinary differential equation

$$\frac{y'(x)}{y(x)}=\frac{d\ln\left(y(x)\right)}{dx}=\alpha(x)+\beta(x)f\left(y(x)\right). \tag{A.3}$$

This differential equation lies at the heart of the functional form question originally posed by Muellbauer (1975, 1976). In particular, the simplest form of this question is, "What is the class of functions $f\left(y\right)$ that can satisfy (A.3) and the 0° homogeneity condition,

$$\alpha'(x)x+\beta'(x)xf(y)+\beta(x)f'(y)y\equiv 0\text{?''} \tag{A.4}$$

It turns out that there are only two possibilities: a special case of Bernoulli's equation,

$$\frac{y'}{y}=\alpha_0+\beta_0\left(\frac{y}{x}\right)^{\kappa},\ \kappa\neq 0; \tag{A.5}$$

or a special case of the logarithmic transformation,

$$\frac{y'}{y}=\alpha_0+\beta_0\ln\left(\frac{y}{x}\right). \tag{A.6}$$

The reason for this can be obtained by analyzing the implications of (A.4) directly. First, consider the case where $\alpha'(x)x=0$, so that $\alpha(x)=\alpha_0$, a constant. Then (A.4) reduces to

$$\beta'(x)xf(y)+\beta(x)f'(y)y\equiv 0, \tag{A.7}$$

or equivalently,

$$\frac{d\ln(f)}{d\ln(y)}=\frac{f'(y)}{f(y)}y=-\frac{\beta'(x)}{\beta(x)}x=-\frac{d\ln(\beta)}{d\ln(x)}=\kappa, \tag{A.8}$$

where κ is a constant because the left-hand-side is independent of $\boldsymbol{x}$, while the right-hand-side is independent of y. Without any loss in generality, the solutions are $f(y) = y^{\kappa}$ and $\beta(x) = \beta_0 x^{-\kappa}$.

Now suppose that $\alpha'(x)x \neq 0$, so that

$$\beta'(x)xf(y) + \beta(x)f'(y)y = -\alpha'(x)x . \tag{A.9}$$

Since the right-hand-side is again independent of y, at least one of the terms on the left also must be independent of y. If $f'(y) = 0$, so that $f(y) = f_0$ is constant, we obtain the degenerate case where the functions of y on the right-hand-side of (A.3) are not linearly independent. Hence, it must be that $\beta'(x)x = 0$, i.e., $\beta(x) = \beta$, a constant, and

$$f'(y)y = \frac{df(y)}{d\ln(y)} = -\frac{\alpha'(x)x}{\beta} = \lambda , \tag{A.10}$$

where λ is a constant again because the left-hand-side is independent of $\boldsymbol{x}$ and the right-hand-side is independent of y. Solving the left side gives

$$f(y) = \lambda \ln(y) + \gamma , \tag{A.11}$$

while the right-hand-side can be rewritten as

$$\frac{d\alpha(x)}{d\ln(x)} = -\lambda\beta , \tag{A.12}$$

which has solution

$$\alpha(x) = \alpha - \lambda\beta\ln(x) . \tag{A.13}$$

Combining (A.11) and (A.13), we obtain (A.6), with $\alpha_0 = \alpha + \beta\gamma$ and $\beta_0 = \beta\lambda$.

The implication is that, for ranks one and two demand models in this class, the admissible forms of $f(y)$ are completely determined by homogeneity.

When we consider incomplete demand systems, we do not have homogeneity in the prices of interest or adding up to restrict the functional form. For Bernoulli's differential equation,

$$y' = \alpha(x)y + \beta(x)y^{1-\kappa}, \ \kappa \neq 0 , \tag{A.14}$$

if we note that $d\left(y^{\kappa}/\kappa\right)/dx = y^{\kappa-1}y'$, we can rewrite this as the linear ordinary differential equation in $f(y) = y^{\kappa}/\kappa$,

$$\frac{d}{dx}\left(y^{\kappa}/\kappa\right) = y^{\kappa-1}y' = \left(\kappa\alpha(x)\right)\left(y^{\kappa}/\kappa\right) + \beta(x) , \tag{A.15}$$

with complete solution

$$y(x)=\left[\kappa e^{\int^x \kappa\alpha(s)ds}\left(\int^x e^{-\int^s \kappa\alpha(t)dt}\beta(s)ds+c\right)\right]^{1/\kappa}. \tag{A.16}$$

Similarly, the logarithmic first-order linear differential equation is

$$\frac{d\ln(y)}{dx}=\frac{y'}{y}=\alpha(x)+\beta(x)\ln\left(y\right), \tag{A.17}$$

with complete solution

$$y(x)=\exp\left\{e^{\int^x \beta(s)ds}\left(\int^x e^{-\int^s \beta(t)dt}\alpha(s)ds+c\right)\right\}. \tag{A.18}$$

The generic nature of both of these differential equations is that they can be written as simple linear first-order ordinary differential equations,

$$\frac{df(y(x))}{dx}=\alpha(x)+\beta(x)f(y(x)). \tag{A.19}$$

When y is normalized income and the demands do not absorb all of the budget, homogeneity and adding up do not impose any restriction on the class of functions $f(y)$ that can solve this differential equation, and the complete class of solutions is

$$y(x)=f^{-1}\left[e^{\int^x \kappa\alpha(s)ds}\left(\int^x e^{-\int^s \kappa\alpha(t)dt}\beta(s)ds+c\right)\right]. \tag{A.20}$$

A.3 Proof of Proposition 1.

Proposition 1*. Every full rank weakly integrable incomplete Gorman system has* $K\le 3$ *and a definition for* $y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)\equiv f\left(e(\boldsymbol{p}(\boldsymbol{x}),\tilde{\boldsymbol{p}},u)\right)$ *exists such that*

$$\frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)}{\partial \boldsymbol{x}}=\begin{cases}\alpha_1(\boldsymbol{x},\tilde{\boldsymbol{p}}) & K=1,\\ \alpha_1(\boldsymbol{x},\tilde{\boldsymbol{p}})+\alpha_2(\boldsymbol{x},\tilde{\boldsymbol{p}})y(\boldsymbol{x},\tilde{\boldsymbol{p}},u) & K=2,\\ \alpha_1(\boldsymbol{x},\tilde{\boldsymbol{p}})+\alpha_2(\boldsymbol{x},\tilde{\boldsymbol{p}})y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)+\alpha_3(\boldsymbol{x},\tilde{\boldsymbol{p}})y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)^2 & K=3.\end{cases}$$

Conversely, if $K\ge 3$ *and a maximum number of the Jacoby brackets,*

$$h_k(y)h_\ell'(y)-h_k'(y)h_\ell(y),\ k<\ell,$$

can be written as linear combinations of the $\{h_1(y)\cdots h_K(y)\}$*,*

$$h_k(y)h_\ell'(y)-h_k'(y)h_\ell(y)=d_{k\ell}^1h_1(y)+\cdots d_{k\ell}^Kh_K(y),\ k<\ell,$$

where the $d_{k\ell}^j$ *are constant, then* $rank\left[\boldsymbol{A}(\boldsymbol{x},\tilde{\boldsymbol{p}})\right]=3$*, and a definition for y exists such that*

$$\frac{\partial y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)}{\partial \boldsymbol{x}} = \alpha_1(\boldsymbol{x},\tilde{\boldsymbol{p}}) + \alpha_2(\boldsymbol{x},\tilde{\boldsymbol{p}})y(\boldsymbol{x},\tilde{\boldsymbol{p}},u) + \cdots + \alpha_K(\boldsymbol{x},\tilde{\boldsymbol{p}})y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)^{K-1}.$$

Proof: By Young's theorem, the second-order cross partial derivatives of y with respect to $\boldsymbol{x}$ must be symmetric for integrability,

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = \sum_{k=1}^{K}\left(\frac{\partial \alpha_{ik}}{\partial x_j}h_k + \alpha_{ik}h_k' \sum_{\ell=1}^{K}\alpha_{j\ell}h_\ell\right)$$

$$= \sum_{k=1}^{K}\left(\frac{\partial \alpha_{jk}}{\partial x_i}h_k + \alpha_{jk}h_k' \sum_{\ell=1}^{K}\alpha_{i\ell}h_\ell\right) = \frac{\partial^2 y}{\partial x_j \partial x_i} \;\; \forall \;\; i \neq j. \qquad \text{(A.21)}$$

These can be re-expressed in terms of ½$n_q(n_q$–1) vanishing differences,

$$0 = \sum_{k=1}^{K}\left(\frac{\partial \alpha_{ik}}{\partial x_j} - \frac{\partial \alpha_{jk}}{\partial x_i}\right)h_k + \sum_{k=1}^{K}\sum_{\ell=1}^{K}\alpha_{ik}\alpha_{j\ell}\left(h_k'h_\ell - h_k h_\ell'\right) \; \forall \; 1 \le j < i = 2,\cdots,n_q. \quad \text{(A.22)}$$

In the double sum on the right-hand-side if (A.22), when $k = \ell$, the term $\alpha_{ik}\alpha_{jk}$ is multiplied by the Jacoby bracket, $h_k'h_k - h_k h_k' = 0$. On the other hand, when $k \neq \ell$, the Jacoby bracket $h_k'h_\ell - h_k h_\ell'$ appears twice, once multiplied by $\alpha_{ik}\alpha_{j\ell}$ and once multiplied by $-\alpha_{i\ell}\alpha_{jk}$. Therefore, rewrite (A.22) as

$$0 = \sum_{k=1}^{K}\left(\frac{\partial \alpha_{ik}}{\partial x_j} - \frac{\partial \alpha_{jk}}{\partial x_i}\right)h_k + \sum_{k=2}^{K}\sum_{\ell=1}^{k-1}\left(\alpha_{ik}\alpha_{j\ell} - \alpha_{jk}\alpha_{i\ell}\right)\left(h_k'h_\ell - h_k h_\ell'\right), 1 \le j < i = 2\cdots n_q, \quad \text{(A.23)}$$

a linear system of ½$n_q(n_q$–1) equations in the ½$K(K$–1) Jacoby brackets $h_k'h_\ell - h_k h_\ell'$.

The first step in the proof of the proposition is to restate (A.23) in matrix form.

Define

$$\boldsymbol{B} = \begin{bmatrix} \alpha_{22}\alpha_{11} - \alpha_{12}\alpha_{21} & \cdots & \alpha_{2k}\alpha_{1\ell} - \alpha_{1k}\alpha_{2\ell} & \cdots & \alpha_{2K}\alpha_{1K-1} - \alpha_{1K}\alpha_{2K-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{i2}\alpha_{j1} - \alpha_{j2}\alpha_{i1} & \cdots & \alpha_{ik}\alpha_{j\ell} - \alpha_{ik}\alpha_{j\ell} & \cdots & \alpha_{2K}\alpha_{1K-1} - \alpha_{1K}\alpha_{2K-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n_q 2}\alpha_{n_q-1,1} - \alpha_{n_q-1,2}\alpha_{n_q 1} & \cdots & \alpha_{n_q k}\alpha_{n_q-1,\ell} - \alpha_{n_q-1,k}\alpha_{n_q \ell} & \cdots & \alpha_{n_q K}\alpha_{n_q-1,K-1} - \alpha_{n_q-1,K}\alpha_{n_q K-1} \end{bmatrix},$$

$$\boldsymbol{C} = -\begin{bmatrix} \frac{\partial \alpha_{11}}{\partial x_2} - \frac{\partial \alpha_{21}}{\partial x_1} & \cdots & \frac{\partial \alpha_{1K}}{\partial x_2} - \frac{\partial \alpha_{2K}}{\partial x_1} \\ \vdots & \vdots & \vdots \\ \frac{\partial \alpha_{i1}}{\partial x_j} - \frac{\partial \alpha_{j1}}{\partial x_i} & \cdots & \frac{\partial \alpha_{iK}}{\partial x_j} - \frac{\partial \alpha_{jK}}{\partial x_i} \\ \vdots & \vdots & \vdots \\ \frac{\partial \alpha_{n_q 1}}{\partial x_{n_q - 1}} - \frac{\partial \alpha_{n_q - 1,1}}{\partial x_{n_q}} & \cdots & \frac{\partial \alpha_{n_q K}}{\partial x_{n_q - 1}} - \frac{\partial \alpha_{n_q - 1K}}{\partial x_{n_q}} \end{bmatrix},$$

$$\boldsymbol{h} = \begin{bmatrix} h_1 & \cdots & h_K \end{bmatrix}^\mathsf{T},$$

and
$$\tilde{\boldsymbol{h}} = \begin{bmatrix} h_2' h_1 - h_2 h_1' & \cdots & h_k' h_\ell - h_k h_\ell' & \cdots & h_K' h_{K-1} - h_K h_{K-1}' \end{bmatrix}^\mathsf{T}.$$

B is $\frac{1}{2} n_q (n_q - 1) \times \frac{1}{2} K(K-1)$, **C** is $\frac{1}{2} n_q (n_q - 1) \times K$, **h** is $K \times 1$, and $\tilde{\boldsymbol{h}}$ is $\frac{1}{2} K(K-1) \times 1$.

This gives the symmetry conditions in matrix form as

$$\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{C}\boldsymbol{h}. \tag{A.24}$$

For this to be a well-posed system of equations we must have at least as many equations as unknowns, which is equivalent to $n_q \geq K$. Assume this is so. Premultiply both sides of (A.24) by $\boldsymbol{B}^\mathsf{T}$ to get the square system, $\boldsymbol{B}^\mathsf{T}\boldsymbol{B}\tilde{\boldsymbol{h}} = \boldsymbol{B}^\mathsf{T}\boldsymbol{C}\boldsymbol{h}$. The rank result of Lie (1880) when **B** has full column rank is that $\frac{1}{2} K(K-1) \leq K$, equivalently, $K \leq 3$ (Hermann 1975: 143-146). The reason is a direct result of linear algebra. The rank of ***B*** is inherited from the rank of a (Hermann 1975: 141). Since $\boldsymbol{B}^\mathsf{T}\boldsymbol{B}$ is of order $\frac{1}{2} K(K-1) \times \frac{1}{2} K(K-1)$ and has rank no greater than K (the rank of ***A***), it follows that $K \leq 3$, completing the proof of the first part of the proposition for the full rank case.

The next step is to obtain the representation result for the full rank case. Assume that ***B*** has full column rank. The least squares formula for $\tilde{\boldsymbol{h}}$ as a function of ***h*** is

$$\tilde{\boldsymbol{h}} = (\boldsymbol{B}^\mathsf{T}\boldsymbol{B})^{-1}\boldsymbol{B}^\mathsf{T}\boldsymbol{C}\boldsymbol{h} \equiv \boldsymbol{D}\boldsymbol{h}. \tag{A.25}$$

The vectors $\tilde{\boldsymbol{h}}$ and ***h*** depend only on y and not on $(\boldsymbol{x}, \tilde{\boldsymbol{p}})$, while the matrix ***D*** depends only on $(\boldsymbol{x}, \tilde{\boldsymbol{p}})$ and not on y. It follows that the elements of ***D*** are absolute constants; a fundamental property that we require below. Since ***B*** is of order $\frac{1}{2} n_q (n_q - 1) \times \frac{1}{2} K(K-1)$ and ***C*** is of order $\frac{1}{2} n_q (n_q - 1) \times K$, it follows that ***D*** is of order $\frac{1}{2} K(K-1) \times K$. That is, when $K = 1$, ***D*** has zero rows (there are no Jacoby brackets), when $K = 2$, ***D*** has one row and two columns), and when $K = 3$, ***D*** has three rows and three columns. If $K > 3$, then ***D*** would have more rows than columns (i.e., more Jacoby brackets than income functions), and the full rank condition cannot be satisfied. We address each full rank case in turn.

Rank 1:
$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha_1 h_1(y)\,. \tag{A.26}$$

Integrability implies that

$$\frac{\partial^2 y}{\partial \boldsymbol{x} \partial \boldsymbol{x}^{\top}} = \frac{\partial \alpha_1}{\partial \boldsymbol{x}^{\top}} h_1(y) + h_1'(y)\alpha_1\alpha_1^{\top} \tag{A.27}$$

is symmetric. Hence, $\partial\alpha_1 / \partial \boldsymbol{x}^{\top}$ must be symmetric, which is necessary and sufficient for the existence of a function, $\beta : \mathscr{P} \times \tilde{\mathscr{P}} \to \mathbb{R}$, such that $\partial\beta/\partial\boldsymbol{x} = \alpha_1$. Rewrite the demands as

$$\frac{\partial y}{\partial \boldsymbol{x}} = \frac{\partial \beta}{\partial \boldsymbol{x}} h_1(y)\,, \tag{A.28}$$

and separate the variables (recall that $h_1(y) \neq 0$ is required for $\partial y/\partial \boldsymbol{x} \gg \boldsymbol{0}$) to obtain

$$\gamma(y) \equiv \int^{y} h_1(s)^{-1} ds = \beta(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \theta(\tilde{\boldsymbol{p}}, u)\,. \tag{A.29}$$

From this we have

$$\frac{\partial \gamma}{\partial \boldsymbol{x}} = \gamma'(y)\frac{\partial y}{\partial \boldsymbol{x}} = \frac{1}{h_1(y)}\frac{\partial y}{\partial \boldsymbol{x}} = \alpha_1\,. \tag{A.30}$$

Therefore, a representation for y exists (by composing γ and f), such that $\partial y/\partial \boldsymbol{x} = \alpha_1$ and $y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \beta(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \theta(\tilde{\boldsymbol{p}}, u)$, with $\alpha_1 = \partial\beta(\boldsymbol{x}, \tilde{\boldsymbol{p}})/\partial \boldsymbol{x}$.

Rank 2:
$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha_1 h_1(y) + \alpha_2 h_2(y)\,. \tag{A.31}$$

Integrability implies that

$$\frac{\partial^2 y}{\partial \boldsymbol{x} \partial \boldsymbol{x}^{\top}} = \frac{\partial \alpha_1}{\partial \boldsymbol{x}^{\top}} h_1 + \frac{\partial \alpha_2}{\partial \boldsymbol{x}^{\top}} h_2 + \left(\alpha_1 h_1' + \alpha_2 h_2'\right)\left(\alpha_1 h_1 + \alpha_2 h_2\right)^{\top} \tag{A.32}$$

is symmetric. Expanding gives

$$\frac{\partial^2 y}{\partial \boldsymbol{x} \partial \boldsymbol{x}^{\top}} = \frac{\partial \alpha_1}{\partial \boldsymbol{x}^{\top}} h_1 + \frac{\partial \alpha_2}{\partial \boldsymbol{x}^{\top}} h_2 + \alpha_1\alpha_1^{\top} h_1' h_1 + \alpha_1\alpha_2^{\top} h_1' h_2 + \alpha_2\alpha_1^{\top} h_2' h_1 + \alpha_2\alpha_2^{\top} h_2' h_2\,, \tag{A.33}$$

and the terms $\alpha_1\alpha_1^{\top} h_1' h_1$ and $\alpha_2\alpha_2^{\top} h_2' h_2$ are automatically symmetric. Since α_1 and α_2 are linearly independent, $\alpha_2 \neq c\alpha_1$ for any $c \in \mathbb{R}$. Otherwise, the rank of $\boldsymbol{A}(\boldsymbol{x}, \tilde{\boldsymbol{p}}) = [\alpha_1\, \alpha_2]$ is only 1, not 2. Hence, $\alpha_1\alpha_2^{\top}$ is not symmetric. Since h_1 and h_2 are functionally independent (equivalently, are locally linearly independent), $h_1' h_2 \neq h_1 h_2'$. Otherwise, $h_2 = c h_1$ for some constant $c \in \mathbb{R}$; a contradiction. Hence, we can premultiply the reduced symmetry conditions by $\alpha_1^\top$ and postmultiply by α_2 to obtain

$$\alpha_1^\top\left(\frac{\partial\alpha_1}{\partial \boldsymbol{x}^\top}h_1+\frac{\partial\alpha_2}{\partial \boldsymbol{x}^\top}h_2+\alpha_1\alpha_2^\top h_1'h_2+\alpha_2\alpha_1^\top h_2'h_1\right)\alpha_2=$$

$$\alpha_1^\top\frac{\partial\alpha_1}{\partial \boldsymbol{x}^\top}\alpha_2 h_1+\alpha_1^\top\frac{\partial\alpha_2}{\partial \boldsymbol{x}^\top}\alpha_2 h_2+\alpha_1^\top\alpha_1\alpha_2^\top\alpha_2 h_1'h_2+(\alpha_1^\top\alpha_2)^2 h_2'h_1=$$

$$\alpha_i^\top\alpha_j\ \alpha_1^\top\frac{\partial\alpha_1^\top}{\partial \boldsymbol{x}}\alpha_2 h_1+\alpha_1^\top\frac{\partial\alpha_2^\top}{\partial \boldsymbol{x}}\alpha_2 h_2+(\alpha_1^\top\alpha_2)^2 h_1'h_2+\alpha_1^\top\alpha_1\alpha_2^\top\alpha_2 h_2'h_1= \tag{A.34}$$

$$\alpha_1^\top\left(\frac{\partial\alpha_1^\top}{\partial \boldsymbol{x}}h_1+\frac{\partial\alpha_2^\top}{\partial \boldsymbol{x}}h_2+\alpha_2\alpha_1^\top h_1'h_2+\alpha_1\alpha_2^\top h_2'h_1\right)\alpha_2 .$$

Group common terms in the $h_k h_\ell'$ and the h_k and rearrange to write

$$\left[\alpha_1^\top\alpha_1\alpha_2^\top\alpha_2-(\alpha_1^\top\alpha_2)^2\right](h_1h_2'-h_1'h_2)=\alpha_1^\top\left(\frac{\partial\alpha_1}{\partial \boldsymbol{x}^\top}-\frac{\partial\alpha_1^\top}{\partial \boldsymbol{x}}\right)\alpha_2 h_1+\alpha_1^\top\left(\frac{\partial\alpha_2}{\partial \boldsymbol{x}^\top}-\frac{\partial\alpha_2^\top}{\partial \boldsymbol{x}}\right)\alpha_2 h_2 . \tag{A.35}$$

Solving for the Jacoby bracket, $h_1h_2'-h_1'h_2$, we have

$$h_1h_2'-h_1'h_2=\left[\frac{\alpha_1^\top\left(\partial\alpha_1/\partial \boldsymbol{x}^\top-\partial\alpha_1^\top/\partial \boldsymbol{x}\right)\alpha_2}{\alpha_1^\top\alpha_1\alpha_2^\top\alpha_2-(\alpha_1^\top\alpha_2)^2}\right]h_1+\left[\frac{\alpha_1^\top\left(\partial\alpha_2/\partial \boldsymbol{x}^\top-\partial\alpha_2^\top/\partial \boldsymbol{x}\right)\alpha_2}{\alpha_1^\top\alpha_1\alpha_2^\top\alpha_2-(\alpha_1^\top\alpha_2)^2}\right]h_2$$

$$\equiv c_1h_1+c_1h_2 , \tag{A.36}$$

with c_1 and c_2 constants, both of which cannot vanish. Without any loss in generality, let $h_1\neq 0$ (both h_i cannot vanish simultaneously and neither can vanish over an open set). Dividing both sides of (A.36) by h_1 and solving for h_2' gives

$$h_2'(y)=c_1-\frac{h_1'(y)}{h_1(y)}+\frac{c_2}{h_1(y)}h_2(y) . \tag{A.37}$$

Let $c_1\neq 0$ (reverse the roles of h_1 and h_2, if necessary) and make a change of variables to $\tilde{h}_1=c_1h_1+c_1h_2$, with $\tilde{h}_1'=c_1h_1'+c_1h_2'$, and to $\tilde{h}_2=h_2/c_1$, with $\tilde{h}_2'=h_2'/c_1$. Then

$$\tilde{h}_1\tilde{h}_2'-\tilde{h}_1'\tilde{h}_2=(c_1h_1+c_2h_2)(h_2'/c_1)-(c_1h_1'+c_2h_2')(h_2/c_1)=h_1h_2'-h_1'h_2 . \tag{A.38}$$

We now have

$$\tilde{h}_1\tilde{h}_2' - \tilde{h}_1'\tilde{h}_2 = h_1h_2' - h_1'h_2 = c_1h_1 + c_2h_2 = \tilde{h}_1 . \tag{A.39}$$

In other words (abusing notation by dropping the ~'s), we form particular linear combinations of the h_i such that

$$h_1h_2' - h_1'h_2 = h_1 \neq 0 , \tag{A.40}$$

equivalently,

$$h_2' - \frac{h_1'}{h_1}h_2 = 1 . \tag{A.41}$$

Since

$$\frac{d}{dy}\left(\frac{h_2}{h_1}\right) = \frac{h_2'}{h_1} - \frac{h_1'}{h_1^2}h_2 = \frac{1}{h_1} , \tag{A.42}$$

direct integration gives

$$\int \frac{d}{dy}\left(\frac{h_2(y)}{h_1(y)}\right)dy = \int \frac{dy}{h_1(y)} , \tag{A.43}$$

equivalently,

$$h_2(y) = h_1(y)\int^y \frac{ds}{h_1(s)} . \tag{A.44}$$

Define $\gamma(y) = \int^y h_1(s)^{-1}ds$ and rewrite (A.31) as

$$\frac{\partial y}{\partial \boldsymbol{x}} = \left[\alpha_1 + \alpha_2\gamma(y)\right]h_1(y) . \tag{A.45}$$

Since $\gamma'(y) = \frac{d}{dy}\int^y h_1(s)^{-1}ds = h_1(y)^{-1}$, this is equivalent to

$$\frac{\partial \gamma}{\partial \boldsymbol{x}} = \gamma'(y)\frac{\partial y}{\partial \boldsymbol{x}} = \frac{1}{h_1(y)}\frac{\partial y}{\partial \boldsymbol{x}} = \alpha_1 + \alpha_2\gamma(y) . \tag{A.46}$$

We thus can change the definition of $y = f(m)$ to incorporate $\gamma(y)$ through composition, and any full rank 2 system has a representation for y such that $\partial y/\partial \boldsymbol{x} = \alpha_1 + \alpha_2 y$.

Rank 3:

$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha_1h_1(y) + \alpha_2h_2(y) + \alpha_3h_3(y) \tag{A.47}$$

The derivations in this case are considerably more involved. We make use of several previous results and techniques from the theory of differential equations to simplify and reduce the calculations. Let $h_1(y) \neq 0$, define $\gamma(y) = \int^y h_1(s)^{-1} ds$, and rewrite (A.47) as

$$\frac{\partial \gamma}{\partial \boldsymbol{x}} = \gamma'(y) \frac{\partial y}{\partial \boldsymbol{x}} = \frac{1}{h_1(y)} \frac{\partial y}{\partial \boldsymbol{x}} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 \left(\frac{h_2(y)}{h_1(y)} \right) + \boldsymbol{\alpha}_3 \left(\frac{h_3(y)}{h_1(y)} \right)$$

$$\equiv \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 \tilde{h}_2(y) + \boldsymbol{\alpha}_3 \tilde{h}_3(y) \qquad \text{(A.48)}$$

By Lemma 1 symmetry is coordinate free. Therefore, consider the representation (again dropping the ~'s and redefining *y* if necessary)

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 h_2(y) + \boldsymbol{\alpha}_3 h_3(y). \qquad \text{(A.49)}$$

The least squares conversion of the symmetry conditions gives

$$h_2'(y) = c_{12}^1 + c_{12}^2 h_2(y) + c_{12}^3 h_3(y),$$

$$h_3'(y) = c_{13}^1 + c_{13}^2 h_2(y) + c_{13}^3 h_3(y), \qquad \text{(A.50)}$$

$$h_2(y) h_3'(y) - h_2'(y) h_3(y) = c_{23}^1 + c_{23}^2 h_2(y) + c_{23}^3 h_3(y),$$

where the c_{ij}^k are constants and cannot all be zero in any given equation. The first two equations form a complete system of linear, ordinary differential equations with constant coefficients. These would be straightforward to solve if the system were not constrained by the third equation (the Jacoby bracket for $h_2(y)$ and $h_3(y)$).

Our plan of attack is to calculate the complete solution to the two-equation system of differential equations and then check for consistency with the third equation. This second step restricts the set of values that the c_{ij}^k can assume in an integrable system. Differentiate the first equation with respect to *y* and substitute out $h_3'(y)$ and $h_3(y)$,

$$h_2''(y) = c_{12}^2 h_2'(y) + c_{12}^3 h_3'(y)$$

$$= c_{12}^2 h_2'(y) + c_{12}^3 \left[c_{13}^1 + c_{13}^2 h_2(y) + c_{13}^3 h_3(y) \right]$$

$$= c_{12}^3 c_{13}^1 + c_{12}^2 h_2'(y) + c_{12}^3 c_{13}^2 h_2(y) + c_{12}^3 c_{13}^3 h_3(y) \qquad \text{(A.51)}$$

$$= c_{12}^3 c_{13}^1 + c_{12}^2 h_2'(y) + c_{12}^3 c_{13}^2 h_2(y) + c_{13}^3 \left[h_2'(y) - c_{12}^1 + c_{12}^2 h_2(y) \right]$$

$$= \left(c_{12}^2 + c_{13}^3 \right) h_2'(y) + \left(c_{12}^3 c_{13}^2 - c_{12}^2 c_{13}^3 \right) h_2(y) + \left(c_{13}^1 c_{12}^3 - c_{12}^1 c_{13}^3 \right).$$

The homogeneous part of this second-order differential equation is

$$h_2''(y) - \left(c_{12}^2 + c_{13}^3\right)h_2'(y) - \left(c_{12}^3 c_{13}^2 - c_{12}^2 c_{13}^3\right)h_2(y) = 0\,. \tag{A.52}$$

Trying $h_2(y) = e^{\lambda y}$ produces the characteristic equation

$$\lambda^2 - \left(c_{12}^2 + c_{13}^3\right)\lambda - \left(c_{12}^3 c_{13}^2 - c_{12}^2 c_{13}^3\right) = 0\,, \tag{A.53}$$

with characteristic roots

$$\lambda = ½\left[c_{12}^2 + c_{13}^3 \pm \sqrt{\left(c_{12}^2 + c_{13}^3\right)^2 + 4\left(c_{12}^3 c_{13}^2 - c_{12}^2 c_{13}^3\right)}\right]. \tag{A.54}$$

If $c_{12}^2 = c_{12}^3 = c_{13}^3 = 0$, then $\lambda = 0$ is the only root, and the complete solution has the form

$$h_2(y) = a_2 + b_2 y + c_2 y^2\,,$$

$$h_3(y) = a_3 + b_3 y + c_3 y^2\,, \tag{A.55}$$

where the $\{a_k, b_k, c_k\}_{k=2}^3$ are constants. Define $\tilde{\alpha}_1 = \alpha_1 + a_2\alpha_2 + a_3\alpha_3$, $\tilde{\alpha}_2 = b_2\alpha_2 + b_3\alpha_3$, and $\tilde{\alpha}_3 = c_2\alpha_2 + c_3\alpha_3$. Then we have

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \tilde{\alpha}_1 + \tilde{\alpha}_2 y + \tilde{\alpha}_3 y^2\,. \tag{A.56}$$

The last step in this part of the proof is to show that this is the only possibility in the full rank 3 case. If any of $c_{12}^2 \neq 0$, $c_{12}^3 \neq 0$, or $c_{13}^3 \neq 0$, then we need to consider distinct and repeated roots separately. With distinct roots, the complete solution to the two ordinary differential equations takes the general form

$$h_2(y) = a_2 + b_2 e^{\lambda_1 y} + c_2 e^{\lambda_2 y}\,,$$

$$h_2(y) = a_3 + b_3 e^{\lambda_1 y} + c_3 e^{\lambda_2 y}\,, \tag{A.57}$$

where the $\{a_k, b_k, c_k\}_{k=2}^3$ again are constants. As before, define $\tilde{\alpha}_1 = \alpha_1 + a_2\alpha_2 + a_3\alpha_3$, $\tilde{\alpha}_2 = b_2\alpha_2 + b_3\alpha_3$ and $\tilde{\alpha}_3 = c_2\alpha_2 + c_3\alpha_3$, and rewrite (A.49) as

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \tilde{\alpha}_1 + \tilde{\alpha}_2 e^{\lambda_1 y} + \tilde{\alpha}_3 e^{\lambda_2 y}\,. \tag{A.58}$$

The equation for the Jacoby bracket $h_2 h_3' - h_2' h_3$ now takes the form

$$(\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)y} = c_{23}^1 + c_{23}^2 e^{\lambda_1 y} + c_{23}^3 e^{\lambda_2 y}\,, \tag{A.59}$$

where $\lambda_2 - \lambda_1 = \sqrt{\left(c_{12}^2 + c_{13}^3\right)^2 + 4\left(c_{12}^3 c_{13}^2 - c_{12}^2 c_{13}^3\right)}$ and $\lambda_2 + \lambda_1 = c_{12}^2 + c_{13}^3$; a contradiction for all $(\lambda_1, \lambda_2) \neq (0,0)$, for either real or complex roots. Therefore, the roots must be real and equal, $\lambda = \frac{1}{2}\left(c_{12}^2 + c_{13}^3\right)$. Once again form the above linear combinations of the α_k's, let $h_2(y) = e^{\lambda y}$ and $h_3(y) = ye^{\lambda y}$, and rewrite (A.49) as

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \tilde{\alpha}_1 + \tilde{\alpha}_2 e^{\lambda y} + \tilde{\alpha}_3 y e^{\lambda y}. \tag{A.60}$$

In this case, the equation for the Jacoby bracket, $h_2 h_3' - h_2' h_3$, takes the form

$$e^{2\lambda y} = c_{23}^1 + c_{23}^2 e^{\lambda y} + c_{23}^3 y e^{\lambda y}, \tag{A.61}$$

a contradiction for all $\lambda \neq 0$. Hence, only a repeated vanishing root is possible and a representation for y exists in any full rank 3 system such that

$$\frac{\partial y}{\partial \boldsymbol{x}} \equiv \alpha_1 + \alpha_2 y + \alpha_3 y^2. \tag{A.62}$$

This completes the proof of the full rank representation part of the proposition.

The next step in the proof of the proposition is to show that polynomials constitute the class of minimal deficit demand systems when $K > 3$. This is accomplished by an inductive argument, and we proceed with the induction beginning with $K = 4$. When $K = 4$ there are a total of six Jacoby brackets, but the dimension of the vector space spanned by the basis $\{h_1\, h_2 \cdots h_4\}$ is only four. We know from the theory of Lie algebras on the real line that at least one of the Jacoby brackets must lie outside of this space. We have shown that by redefining y and modifying the α_k's to accommodate the change in y, $\{1\, y\, y^2\}$ is the largest Lie algebra on the real line. The structure of this vector space is

$$\begin{bmatrix} h_2' \\ h_3' \\ h_2 h_3' - h_2' h_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2y \\ y^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}. \tag{A.63}$$

If we add a fourth income function to this system, the above derivations apply with minor modifications. Therefore, add $h_4(y)$ to $\{1\, y\, y^2\}$. At most two of the new equations for the Jacoby bracket can be consistent. Without loss in generality, consider the fourth and fifth symmetry conditions to be

$$h_4'(y) = c_{14}^1 + c_{14}^2 y + c_{14}^3 y^2 + c_{14}^4 h_4(y),$$

$$y h_4'(y) - h_4(y) = c_{24}^1 + c_{24}^2 y + c_{24}^3 y^2 + c_{24}^4 h_4(y). \tag{A.64}$$

The Jacoby bracket conditions then are

$$\begin{bmatrix} h_2' \\ h_3' \\ h_2h_3' - h_2'h_3 \\ h_4' \\ h_2h_4' - h_2'h_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2y \\ y^2 \\ h_4' \\ yh_4' - h_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ c_{14}^1 & c_{14}^2 & c_{14}^2 & c_{14}^3 \\ c_{24}^1 & c_{24}^2 & c_{24}^3 & c_{24}^4 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \\ h_4 \end{bmatrix}. \qquad \text{(A.65)}$$

(A.64) is two linear, first-order ordinary differential equations in $h_4(y)$. The most direct route is to solve the first and check the second for consistency. If $c_{14}^4 = 0$, then integrating the first equation gives

$$h_4(y) = c + c_{14}^1 y + \tfrac{1}{2} c_{14}^2 y^2 + \tfrac{1}{3} c_{14}^3 y^3, \qquad \text{(A.66)}$$

where *c* is a constant of integration. Applying similar modifications to the α_k's as before, we have $h_k(y) = y^{k-1}$, $k = 1,2,3,4$. The second equation becomes

$$yh_4'(y) - h_4(y) = 3y^3 - y^3 = 2y^3 = 0\cdot 1 + 0\cdot y + 0\cdot y^2 + 2y^3, \qquad \text{(A.67)}$$

which is contained in the vector space spanned by $\{1\; y\; y^2\; y^3\}$. Of course, the Jacoby bracket, $h_3h_4' - h_3'h_4 = 3y^4 - 2y^4 = y^4$ falls outside of this vector space, as it must.
If $c_{14}^4 \neq 0$, integrating by parts twice gives the complete solution as

$$h_4(y) = -\left[\frac{c_{14}^1}{c_{14}^4} + \frac{c_{14}^2}{(c_{14}^4)^2} + \frac{2c_{14}^3}{(c_{14}^4)^3}\right] - \left[\frac{c_{14}^2}{c_{14}^4} + \frac{c_{14}^3}{(c_{14}^4)^2}\right] y - \frac{c_{14}^3}{c_{14}^4} y^2 + ce^{c_{14}^4 y}, \qquad \text{(A.68)}$$

where c is again a constant of integration. Once more using the above device to adjust the α_k's, we have $h_4(y) = e^{c_{14}^4 y}$, and the second equation becomes

$$yh_4'(y) - h_4(y) = (c_{14}^4 y - 1)e^{c_{14}^4 y} = c_{24}^1 + c_{24}^2 y + c_{24}^3 y^2 + c_{24}^4 e^{c_{14}^4 y}, \qquad \text{(A.69)}$$

a contradiction. Hence, the structure with four income functions and a maximum number of Jacoby brackets spanned by the income functions $\{1\; y\; y^2\; y^3\}$ is

$$\begin{bmatrix} h_2' \\ h_3' \\ h_4' \\ h_2h_3' - h_2'h_3 \\ h_2h_4' - h_2'h_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2y \\ 3y^2 \\ y^2 \\ 2y^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \\ y^3 \end{bmatrix}. \qquad \text{(A.70)}$$

Only one (the minimal possible number) Jacoby bracket, $h_3h_4' - h_3'h_4$, out of the total of six, falls outside of this vector space.

The induction is completed by identical steps to show that if a basis with K functions is $\{1\,y\,y^2 \cdots y^{K-1}\}$ and we add a K+1st function, $h_{K+1}(y)$, then the maximal increase in the number of spanned Jacoby brackets occurs when $h_{K+1}(y) = y^K$.

The final step in the proof of this proposition is to show that for the polynomial class of Gorman Engel curve systems, $rank[\boldsymbol{A}(\boldsymbol{x},\tilde{\boldsymbol{p}})] \le 3$. We proceed constructively by showing that if the system of demands has the polynomial representation,[18]

$$\frac{\partial y}{\partial \boldsymbol{x}} = \sum_{k=0}^{K} \boldsymbol{\alpha}_k y^k, \tag{A.71}$$

and is weakly integrable, then there exist $\varphi_k : \mathcal{P} \times \tilde{\mathcal{P}} \to \mathbb{R},\ k = 2,\ldots,K$ such that

$$\boldsymbol{\alpha}_k \equiv \varphi_k \boldsymbol{\alpha}_K \ \ \forall\ k \ge 2. \tag{A.72}$$

Integrability is equivalent to symmetry of the matrix

$$\sum_{k=0}^{K} \frac{\partial \boldsymbol{\alpha}_i}{\partial \boldsymbol{x}^{\top}} y^k + \sum_{k=1}^{K}\sum_{\ell=0}^{K} k \boldsymbol{\alpha}_k \boldsymbol{\alpha}_\ell^{\top} y^{k+\ell-1}. \tag{A.73}$$

By continuity, symmetry requires that each like power of y has a symmetric coefficient matrix, and all of the matrices for powers K+1 through 2K–2 involve nontrivial symmetry conditions without involving any of the $\partial \boldsymbol{\alpha}_k / \partial \boldsymbol{x}^{\top}$ terms. The matrix on y^{2K-1} only involves $\boldsymbol{\alpha}_K \boldsymbol{\alpha}_K^{\top}$ and is symmetric. Combine terms in like powers of y and apply a backward recursion beginning with the matrix on y^{2K-2}, so that

$$(K-1)\boldsymbol{\alpha}_{K-1}\boldsymbol{\alpha}_K^{\top} + K\boldsymbol{\alpha}_K \boldsymbol{\alpha}_{K-1}^{\top} \tag{A.74}$$

is symmetric if and only if $\boldsymbol{\alpha}_{K-1} \equiv \varphi_{K-1}\boldsymbol{\alpha}_K$ for some $\varphi_{K-1} : \mathcal{P} \times \tilde{\mathcal{P}} \to \mathbb{R}$. Similarly,

$$(K-2)\boldsymbol{\alpha}_{K-2}\boldsymbol{\alpha}_K^{\top} + (K-1)\boldsymbol{\alpha}_{K-1}\boldsymbol{\alpha}_{K-1}^{\top} + K\boldsymbol{\alpha}_K \boldsymbol{\alpha}_{K-2}^{\top} \tag{A.75}$$

is symmetric if and only if $\boldsymbol{\alpha}_{K-2} \equiv \varphi_{K-2}\boldsymbol{\alpha}_K$ for some $\varphi_{K-2} : \mathcal{P} \times \tilde{\mathcal{P}} \to \mathbb{R}$. Applying the recursive argument, consider the matrix on y^{2K-4},

$$(K-3)\boldsymbol{\alpha}_{K-3}\boldsymbol{\alpha}_K^{\top} + (K-2)\boldsymbol{\alpha}_{K-2}\boldsymbol{\alpha}_{K-1}^{\top} + (K-1)\boldsymbol{\alpha}_{K-1}\boldsymbol{\alpha}_{K-2}^{\top} + K\boldsymbol{\alpha}_K \boldsymbol{\alpha}_{K-3}^{\top}. \tag{A.76}$$

The middle two terms are symmetric, because $\boldsymbol{\alpha}_{K-2}\boldsymbol{\alpha}_{K-1}^{\top} = \varphi_{K-2}\varphi_{K-1}\boldsymbol{\alpha}_K\boldsymbol{\alpha}_K^{\top} = \boldsymbol{\alpha}_{K-1}\boldsymbol{\alpha}_{K-2}^{\top}$. The matrix $\left(\boldsymbol{\alpha}_{K-3}\boldsymbol{\alpha}_K^{\top} + \boldsymbol{\alpha}_K\boldsymbol{\alpha}_{K-3}^{\top}\right)$ is automatically symmetric. Therefore, the matrix on y^{2K-4} is symmetric if and only if $\boldsymbol{\alpha}_K\boldsymbol{\alpha}_{K-3}^{\top}$ is symmetric, if and only if $\boldsymbol{\alpha}_{K-3} \equiv \varphi_{K-3}\boldsymbol{\alpha}_K$,

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[18] *Switching indexes from {1,…,*K*} to {0,…,*K*} greatly simplifies the algebra and notation in this part of the proof without affecting the structure of the underlying problem in any way.*

for some $\varphi_{K-3} : \boldsymbol{\mathcal{P}} \times \tilde{\boldsymbol{\mathcal{P}}} \to \mathbb{R}$. This completes the argument when $3 \le K \le 5$.

If $K > 5$, for each j satisfying $4 \le j \le K\text{–}1$, group like terms, substitute $\alpha_{K-i} \equiv \varphi_{K-i}\alpha_K$ for each $i < j$, and appeal to symmetry of the matrix $\alpha_{K+1-j}\alpha_K^{\mathsf{T}} + \alpha_K\alpha_{K+1-j}^{\mathsf{T}}$. Then symmetry sequentially requires that each matrix of the following form is symmetric:

$$(j-1)\alpha_K\alpha_{K+1-j}^{\mathsf{T}} + \sum_{i=1}^{j-2}(K-i)\varphi_{K-i}\varphi_{K+1+i-j}\alpha_K\alpha_K^{\mathsf{T}} . \tag{A.77}$$

This holds if and only if $\alpha_{K+1-j} \equiv \varphi_{K+1-j}\alpha_K$ for $\varphi_{K+1-j} : \boldsymbol{\mathcal{P}} \times \tilde{\boldsymbol{\mathcal{P}}} \to \mathbb{R}$. When $j = 4$ we have the result for α_{K-3}; when $j = K-1$ we have it for α_2; and $\forall\ K > 2$, we have $\alpha_k \equiv \varphi_k\alpha_K\ \ \forall\ k = 2,\ldots,K$ so that $rank[\boldsymbol{A}(\boldsymbol{x},\tilde{\boldsymbol{p}})] \le 3$. ■

A.4 Characterizing Indirect Preferences

In this section, we characterize the class of indirect preferences for each of the full rank cases and present and discuss an example of indirect preferences that gives rise to a rank three demand model with more than three income terms.

Rank 1:

$$\frac{\partial y}{\partial \boldsymbol{x}} = \frac{\partial \beta}{\partial \boldsymbol{x}} . \tag{A.78}$$

Simply integrating gives

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},u) = \beta(\boldsymbol{x},\tilde{\boldsymbol{p}}) + \theta(\tilde{\boldsymbol{p}},u) . \tag{A.79}$$

This is the translation group representation of indirect preferences for the rank one case. Solving for the normalized expenditure function gives

$$e(\boldsymbol{p},\tilde{\boldsymbol{p}},u) = m\big(\beta(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}}) + \theta(\tilde{\boldsymbol{p}},u)\big) . \tag{A.80}$$

Equivalently, the indirect utility function has the form

$$v(\boldsymbol{p},\tilde{\boldsymbol{p}},m) = \psi\big(f(m) - \beta(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}}),\tilde{\boldsymbol{p}}\big) , \tag{A.81}$$

where ψ is the inverse of θ with respect to u. Since $\boldsymbol{q} = \boldsymbol{diag}[g_i'] \times (\partial\beta/\partial\boldsymbol{x})/f'$, the demands for **q** in the rank 1 case are homothetic with income elasticity $-mf''(m)/f'(m)$. If $f(m) = m^{\kappa}$, the common income elasticity $1\text{–}\kappa$ is constant, but only equals one in the limiting case $f(m) = \ln(m)$. More general transformations do not result in a constant income elasticity, although it must be independent of prices in this class of demands.

Rank 2:

$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha_1 + \alpha_2 y . \tag{A.82}$$

Symmetry in this case implies

$$\frac{\partial^2 y}{\partial \boldsymbol{x}\partial \boldsymbol{x}^{\mathsf{T}}} = \frac{\partial \alpha_1}{\partial \boldsymbol{x}^{\mathsf{T}}} + \frac{\partial \alpha_2}{\partial \boldsymbol{x}^{\mathsf{T}}} y + \alpha_2 \alpha_1^{\mathsf{T}} + \alpha_2 \alpha_2^{\mathsf{T}} y = \frac{\partial \alpha_1^{\mathsf{T}}}{\partial \boldsymbol{x}} + \frac{\partial \alpha_2^{\mathsf{T}}}{\partial \boldsymbol{x}} y + \alpha_1 \alpha_2^{\mathsf{T}} + \alpha_2 \alpha_2^{\mathsf{T}} y . \quad \text{(A.83)}$$

Eliminating the symmetric matrix $\alpha_2 \alpha_2^{\mathsf{T}} y$ from both sides and equating the matrices that multiply like powers of y implies

$$\frac{\partial \alpha_1}{\partial \boldsymbol{x}^{\mathsf{T}}} + \alpha_2 \alpha_1^{\mathsf{T}} = \frac{\partial \alpha_1^{\mathsf{T}}}{\partial \boldsymbol{x}} + \alpha_1 \alpha_2^{\mathsf{T}} , \quad \text{(A.84)}$$

and that $\partial \alpha_2 / \partial \boldsymbol{x}^{\mathsf{T}}$ is symmetric. The latter property implies the existence of a function $\beta : \mathcal{P} \times \tilde{\mathcal{P}} \rightarrow \mathbb{R}$ such that $\partial \beta / \partial \boldsymbol{x} = \alpha_2$. It follows that $\partial \alpha_1 / \partial \boldsymbol{x}^{\mathsf{T}} + (\partial \beta / \partial \boldsymbol{x}) \alpha_1^{\mathsf{T}}$ is symmetric. Equivalently, we can rewrite (A.82) in the form

$$\frac{\partial y}{\partial \boldsymbol{x}} = \alpha_1 + \frac{\partial \beta}{\partial \boldsymbol{x}} y , \quad \text{(A.85)}$$

with

$$\frac{\partial \alpha_1}{\partial \boldsymbol{x}^{\mathsf{T}}} - \alpha_1 \frac{\partial \beta}{\partial \boldsymbol{x}^{\mathsf{T}}} = \frac{\partial \alpha_1^{\mathsf{T}}}{\partial \boldsymbol{x}} - \frac{\partial \beta}{\partial \boldsymbol{x}} \alpha_1^{\mathsf{T}} ,$$

symmetric. We can apply the integrating factor $e^{-\beta}$ by noting that

$$\frac{\partial}{\partial \boldsymbol{x}} \left(y e^{-\beta} \right) = \left(\frac{\partial y}{\partial \boldsymbol{x}} - \frac{\partial \beta}{\partial \boldsymbol{x}} y \right) e^{-\beta} , \quad \text{(A.86)}$$

and

$$\frac{\partial}{\partial \boldsymbol{x}^{\mathsf{T}}} \left(\alpha_1 e^{-\beta} \right) = \left(\frac{\partial \alpha_1}{\partial \boldsymbol{x}^{\mathsf{T}}} - \alpha_1 \frac{\partial \beta}{\partial \boldsymbol{x}^{\mathsf{T}}} \right) e^{-\beta} \quad \text{(A.87)}$$

is symmetric. This is equivalent to the existence of a function $\gamma : \mathcal{P} \times \tilde{\mathcal{P}} \rightarrow \mathbb{R}$ such that $\partial \gamma / \partial \boldsymbol{x} = \alpha_1 e^{-\beta}$, and integrating gives the transformed deflated expenditure function as

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = e^{\beta(\boldsymbol{x}, \tilde{\boldsymbol{p}})} \gamma(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + e^{\beta(\boldsymbol{x}, \tilde{\boldsymbol{p}})} \theta(\tilde{\boldsymbol{p}}, u) . \quad \text{(A.88)}$$

Let $e^{\beta(\boldsymbol{x}, \tilde{\boldsymbol{p}})} \equiv \delta(\boldsymbol{x}, \tilde{\boldsymbol{p}})$, abuse notation and relabel $e^{\beta(\boldsymbol{x}, \tilde{\boldsymbol{p}})} \gamma(\boldsymbol{x}, \tilde{\boldsymbol{p}})$ as $\gamma(\boldsymbol{x}, \tilde{\boldsymbol{p}})$, and rewrite (A.88) in the form

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \gamma(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \delta(\boldsymbol{x}, \tilde{\boldsymbol{p}}) \theta(\tilde{\boldsymbol{p}}, u) . \quad \text{(A.89)}$$

This quasi-linear form is the translation and scaling group representation of indirect preferences in the full rank two case. We can write the normalized expenditure function as

$$e(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = m\left(\gamma(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \delta(\boldsymbol{x}, \tilde{\boldsymbol{p}}) \theta(\tilde{\boldsymbol{p}}, u) \right) , \quad \text{(A.90)}$$

and the indirect utility function as

$$v(\boldsymbol{p},\tilde{\boldsymbol{p}},m)=\psi\left(\frac{f(m)-\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}})}{\delta(\boldsymbol{x},\tilde{\boldsymbol{p}})},\tilde{\boldsymbol{p}}\right). \tag{A.91}$$

Rank 3:

$$\frac{\partial y}{\partial \boldsymbol{x}}\equiv\alpha_1+\alpha_2 y+\alpha_3 y^2. \tag{A.92}$$

We present two equivalent closed form expressions for the solution to this case. One establishes the connection to the projective transformation group. The other applies when (A.92) has a pair of purely complex roots and provides a direct solution for the trigonometric form of indirect preferences.

First, we note that the methods of van Daal and Merkies (1989) for solving integrability of the complete quadratic expenditure system apply without change to our problem. The only difference is that the homogeneity properties they identified do not apply here. Thus, there is no need to reproduce their steps. They show that (A.92) is integrable if and only if there exist functions, $\beta_1,\beta_2,\beta_3:\mathcal{P}\times\tilde{\mathcal{P}}\to\mathbb{R}$, and $\gamma:\mathbb{R}\to\mathbb{R}$ such that

$$\frac{\partial y}{\partial \boldsymbol{x}}=\frac{\partial\beta_1}{\partial \boldsymbol{x}}+\gamma_2(\beta_2)\beta_3\frac{\partial\beta_2}{\partial \boldsymbol{x}}+\frac{\partial\beta_3}{\partial \boldsymbol{x}}\frac{(y-\beta_1)}{\beta_3}+\frac{\partial\beta_2}{\partial \boldsymbol{x}}\frac{(y-\beta_1)^2}{\beta_3}. \tag{A.93}$$

This can be rewritten in the form

$$\frac{\partial}{\partial \boldsymbol{x}}\left(\frac{y-\beta_1}{\beta_3}\right)=\frac{1}{\beta_3}\left(\frac{\partial y}{\partial \boldsymbol{x}}-\frac{\partial\beta_1}{\partial \boldsymbol{x}}\right)-\frac{(y-\beta_1)}{\beta_3^2}\frac{\partial\beta_3}{\partial \boldsymbol{x}}=\left[\gamma_2(\beta_2)+\frac{(y-\beta_1)^2}{\beta_3^2}\right]\frac{\partial\beta_2}{\partial \boldsymbol{x}}. \tag{A.94}$$

We can simplify this even further by making two simple changes of variables. First, let $w=(y-\beta_1)/\beta_3$, so that

$$\frac{\partial w}{\partial \boldsymbol{x}}=\left[\gamma_2(\beta_2)+w^2\right]\frac{\partial\beta_2}{\partial \boldsymbol{x}} \tag{A.95}$$

Second, let $z=-1/w$, so that

$$\frac{\partial z}{\partial \boldsymbol{x}}=\left[1+\gamma_2(\beta_2)z^2\right]\frac{\partial\beta_2}{\partial \boldsymbol{x}}. \tag{A.96}$$

Now, if $\gamma(\beta_2)\equiv\lambda$ is constant, then we can separate the variables so that

$$\frac{dz}{1+\lambda z^2}=\frac{\partial\beta_2}{\partial x_i}\ \forall\ i=1,\cdots,n_q. \tag{A.97}$$

This is an exact system of partial differential equations and the solution is found by direct integration,

$$\phi\left(\frac{-\beta_3}{y-\beta_1}\right) \equiv \int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1+\lambda z^2)} = \beta_2 + \theta, \tag{A.98}$$

where $\theta(\tilde{\boldsymbol{p}}, u)$ is the "constant of integration." This is readily recognized as the solution obtained by van Daal and Merkies (1989) and applied by Lewbel (1990) to full rank three QPIGL and QPIGLOG complete systems.

But we can go considerably further. One reason for doing this is to obtain closed form expressions for the indirect preferences. a second reason is to show the connection between this representation and the projective transformation group representation of indirect preferences that is standard in the theory of Lie groups. a third reason is that in one case we obtain the trigonometric form of indirect preferences that is implied by Gorman (1981) and is presented without derivation in Lewbel (1990), but heretofore has not been obtained explicitly from the structure of a set of demand equations with complex roots.

Suppose that $\lambda > 0$, define $\lambda = -(\iota\kappa)^2 = \kappa^2$, with $\iota = \sqrt{-1}$, let $\tau = \iota\kappa$ be a purely complex constant, write $1+\lambda z^2 = (1+\tau z)(1-\tau z)$ and apply the method of partial fractions to obtain

$$\frac{1}{1-(\tau z)^2} = \frac{½}{(1-\tau z)} + \frac{½}{(1+\tau z)}. \tag{A.99}$$

This implies that

$$\int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1+\lambda z^2)} = ½\int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1-\tau z)} + ½\int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1+\tau z)}$$

$$= ½\ln\left[\frac{1+\tau\left(\dfrac{-\beta_3}{y-\beta_1}\right)}{1-\tau\left(\dfrac{-\beta_3}{y-\beta_1}\right)}\right] = \beta_2 + \theta. \tag{A.100}$$

Exponentiating and abusing notation by relabeling $e^{2\beta_2}$ as β_2 and $e^{2\theta}$ as θ gives,

$$\frac{y-\beta_1-\tau\beta_3}{y-\beta_1+\tau\beta_3} = \beta_2\theta. \tag{A.101}$$

Solving for the normalized and transformed expenditure function then gives

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \beta_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \tau\beta_3(\boldsymbol{x}, \tilde{\boldsymbol{p}})\left(\frac{1+\beta_2(\boldsymbol{x}, \tilde{\boldsymbol{p}})\theta(\tilde{\boldsymbol{p}}, u)}{1-\beta_2(\boldsymbol{x}, \tilde{\boldsymbol{p}})\theta(\tilde{\boldsymbol{p}}, u)}\right). \tag{A.102}$$

This is an element of the complex projective transformation group in θ. Alternatively, solving for the quasi-indirect utility function,

$$\theta = \upsilon(\boldsymbol{x}, \tilde{\boldsymbol{p}}, y) = \frac{y - \beta_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) - \tau\beta_3(\boldsymbol{x}, \tilde{\boldsymbol{p}})}{\beta_2(\boldsymbol{x}, \tilde{\boldsymbol{p}})\left[y - \beta_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \tau\beta_3(\boldsymbol{x}, \tilde{\boldsymbol{p}})\right]}. \tag{A.103}$$

In this case, (A.103) is an element of the complex projective transformation group in y.

In the case where $\lambda > 0$, we also can derive an alternative, but equivalent, expression for the indirect preferences by using a third change of variables to $s = \kappa z$, where $\lambda = \kappa^2 > 0$, so that

$$\int^{-\beta_3/(y-\beta_1)} \frac{dz}{1+(\kappa z)^2} = \int^{-\kappa\beta_3/(y-\beta_1)} \frac{ds}{\kappa(1+s^2)} = \frac{1}{\kappa}\tan^{-1}\left(\frac{-\kappa\beta_3}{y-\beta_1}\right) = \beta_2 + \theta. \tag{A.104}$$

The indirect utility function therefore can be written as

$$v(\boldsymbol{p}, \tilde{\boldsymbol{p}}, m) = \psi\left\{\frac{1}{\kappa}\tan^{-1}\left(\frac{-\kappa\beta_3(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}})}{f(m) - \beta_1(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}})}\right) - \beta_2(\boldsymbol{g}(\boldsymbol{p}), \tilde{\boldsymbol{p}}), \tilde{\boldsymbol{p}}\right\} \tag{A.105}$$

Alternatively, the normalized and transformed expenditure function can be written as

$$y(\boldsymbol{x}, \tilde{\boldsymbol{p}}, u) = \beta_1(\boldsymbol{x}, \tilde{\boldsymbol{p}}) - \frac{\kappa\beta_3(\boldsymbol{x}, \tilde{\boldsymbol{p}})}{\tan\left\{\kappa\left[\beta_2(\boldsymbol{x}, \tilde{\boldsymbol{p}}) + \theta(\tilde{\boldsymbol{p}}, u)\right]\right\}}. \tag{A.106}$$

Clearly, (42)–(A.106) have the associated trigonometric form for indirect preferences.

Now assume $\lambda < 0$, define $-\lambda = \kappa^2$, and write $1 + \lambda z^2 = (1+\kappa z)(1-\kappa z)$, so that partial fractions imply

$$\frac{1}{1+\lambda z^2} = \frac{1}{1-(\kappa z)^2} = \frac{½}{(1-\kappa z)} + \frac{½}{(1+\kappa z)}. \tag{A.107}$$

Integrating as before now gives

$$\int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1+\lambda z^2)} = ½\int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1-\kappa z)} + ½\int^{-\beta_3/(y-\beta_1)} \frac{dz}{(1+\kappa z)}$$

$$= ½\ln\left[\frac{1+\kappa\left(\frac{-\beta_3}{y-\beta_1}\right)}{1-\kappa\left(\frac{-\beta_3}{y-\beta_1}\right)}\right] = \beta_2 + \theta. \tag{A.108}$$

Notice, in particular, that the only difference between (A.100) and (45) is the purely complex root τ and the purely real root κ, respectively. Therefore, proceeding precisely as before, we obtain (A.102) and (A.103) with κ simply replacing τ everywhere. Hence, the class of indirect utility functions in both cases is

$$v(\boldsymbol{p},\tilde{\boldsymbol{p}},m)=\psi\left\{\frac{f(m)-\beta_1(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}})-\kappa\beta_3(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}})}{\beta_2(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}})\left[f(m)-\beta_1(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}})+\kappa\beta_3(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}})\right]},\tilde{\boldsymbol{p}}\right\}, \quad (A.109)$$

where ψ is once again the inverse of θ with respect to y, and where κ is either purely real or purely complex. The part of v that is associated with (**p**, m) is an element of the (either real or complex) projective transformation group over $y=f(m)$.

To finalize our characterization and exposition, without loss in generality, we can abuse notation further by relabeling $\beta_1+\kappa\beta_3$ as β_1 and $(-\beta_1+\kappa\beta_3)\beta_2$ as β_3, for κ either real or complex. Then we have the three–parameter relationship

$$\theta=\frac{y-\beta_1}{\beta_2 y+\beta_3} \iff y=\frac{\beta_1+\beta_3\theta}{1-\beta_2\theta}. \quad (A.110)$$

Thus, the closed form solutions that can be found in all full rank three cases are members of the projective transformation group. The quasi-indirect utility function, θ, is the inverse group transformation of the (normalized and transformed) expenditure function, *y*. No additional flexibility in income or prices is obtained with a complex κ even though the trigonometric form in (A.106) is an interesting case. Therefore, for the rest of this section, we assume that κ is real.

When κ is real, the space of all projective transformation groups is referred to in differential topology as special linear group two and is denoted by $\mathfrak{sl}(2)$. It is standard practice in Lie group theory to identify the space $\mathfrak{sl}(2)$ with the set of 2×2 real matrices

$$\boldsymbol{A}=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$$

with unit determinant, $\alpha\delta-\beta\gamma=1$. Indeed, we have

$$\boldsymbol{A}^{-1}=\begin{bmatrix}\delta & -\beta\\ -\gamma & \alpha\end{bmatrix}$$

as a member of this group, and if we write

$$y=\frac{\alpha\theta+\beta}{\gamma\theta+\delta} \iff \theta=\frac{\delta y-\beta}{-\gamma y+\alpha}, \quad (A.111)$$

we can see immediately that 2×2 matrix inverses in this set are one-to-one and onto with the inverse functions of the projective transformation group, while $\mathbf{I}_2$ defines the identity map in both spaces. Simple algebra then shows that

$$\frac{\partial y}{\partial \boldsymbol{x}}=\left(\alpha\frac{\partial\beta}{\partial \boldsymbol{x}}-\beta\frac{\partial\alpha}{\partial \boldsymbol{x}}\right)+\left[\left(\beta\frac{\partial\gamma}{\partial \boldsymbol{x}}-\gamma\frac{\partial\beta}{\partial \boldsymbol{x}}\right)-\left(\alpha\frac{\partial\delta}{\partial \boldsymbol{x}}-\delta\frac{\partial\alpha}{\partial \boldsymbol{x}}\right)\right]y+\left(\gamma\frac{\partial\delta}{\partial \boldsymbol{x}}-\delta\frac{\partial\gamma}{\partial \boldsymbol{x}}\right)y^2 . \quad (A.112)$$

The usefulness of this representation is that integrablity is represented clearly, concisely, and simply in the form of four Jacoby brackets between the {α,β,γ,δ} functions with respect to **x**. A (very large) set of full rank three indirect utility functions generating members of Gorman's class of functionally separable demand systems is therefore given by

$$v(\boldsymbol{p},\tilde{\boldsymbol{p}},m)=\upsilon\left\{\frac{\delta(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}})f(m)-\beta(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}})}{-\gamma(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}})f(m)+\alpha(\boldsymbol{g}(\boldsymbol{p}),\tilde{\boldsymbol{p}})},\tilde{\boldsymbol{p}}\right\},\ \alpha\delta-\beta\gamma\equiv 1. \qquad \text{(A.113)}$$

Equivalently, we can represent this class of preference systems in terms of the normalized and transformed expenditure function,

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)=\frac{\alpha(\boldsymbol{x},\tilde{\boldsymbol{p}})\theta(\tilde{\boldsymbol{p}},u)+\beta(\boldsymbol{x},\tilde{\boldsymbol{p}})}{\gamma(\boldsymbol{x},\tilde{\boldsymbol{p}})\theta(\tilde{\boldsymbol{p}},u)+\delta(\boldsymbol{x},\tilde{\boldsymbol{p}})},\ \alpha\delta-\beta\gamma\equiv 1. \qquad \text{(A.114)}$$

Finally, it is worth noting that $\gamma\neq 0$ is required for a full rank three system, and we can define this class of preferences in terms of Lie's (1880) rank three transformation group,

$$y(\boldsymbol{x},\tilde{\boldsymbol{p}},u)=\frac{\tilde{\alpha}(\boldsymbol{x},\tilde{\boldsymbol{p}})\theta(\tilde{\boldsymbol{p}},u)+\tilde{\beta}(\boldsymbol{x},\tilde{\boldsymbol{p}})}{\theta(\tilde{\boldsymbol{p}},u)+\tilde{\delta}(\boldsymbol{x},\tilde{\boldsymbol{p}})} \qquad \text{(A.115)}$$

where $\tilde{\alpha}=\alpha/\gamma$, $\tilde{\beta}=\beta/\gamma$, and $\tilde{\delta}=\delta/\gamma$. We therefore have established the equivalence of all full rank three incomplete Gorman systems, the rank three transformation group of Lie (1880), and the projective transformation group $\mathfrak{sl}(2)$ using elementary methods. ■

A.6 Semidefinite Matrices

Lemma 4. *Let the n×n real-valued matrix **A** be symmetric and positive semidefinite. Then* $\forall\ \boldsymbol{x}\in\mathbb{R}^n:\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x}>0$*, the matrix* $\boldsymbol{A}-(\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x})^{-1}\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}^\top\boldsymbol{A}$ *is symmetric and positive semidefinite, with **x** contained in its null space.*

Proof: Since $\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x}>0$ by hypothesis, $\forall\ \boldsymbol{z}\in\mathbb{R}^n$, $\boldsymbol{z}^\top\left[\boldsymbol{A}-(\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x})^{-1}\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}^\top\boldsymbol{A}\right]\boldsymbol{z}\geq 0$ if and only if $\boldsymbol{z}^\top\boldsymbol{A}\boldsymbol{z}(\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x})\geq(\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{z})^2$. Let the matrix **Q** satisfy $\boldsymbol{A}=\boldsymbol{Q}\boldsymbol{Q}^\top$ and define $\boldsymbol{v}=\boldsymbol{Q}^\top\boldsymbol{z}$ and $\boldsymbol{w}=\boldsymbol{Q}^\top\boldsymbol{x}$. Then $\boldsymbol{z}^\top\boldsymbol{A}\boldsymbol{z}(\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x})\geq(\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{z})^2$ if and only if $\boldsymbol{v}^\top\boldsymbol{v}(\boldsymbol{w}^\top\boldsymbol{w})\geq(\boldsymbol{v}^\top\boldsymbol{w})^2$. The latter is an n–dimensional statement of the Cauchy-Schwarz inequality, and this inequality continues to apply when some of the elements of **v** and or **w** vanish, which can occur if **A** has less than full rank. Finally, inspection verifies that

$$\left[\boldsymbol{A}-(\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x})^{-1}\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}^\top\boldsymbol{A}\right]\boldsymbol{x}=\boldsymbol{A}\boldsymbol{x}-\boldsymbol{A}\boldsymbol{x}=\boldsymbol{0},$$

so that $\boldsymbol{x}$ is contained in the null space of the matrix

$$\left[\boldsymbol{A}-(\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x})^{-1}\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}^\top\boldsymbol{A}\right].$$

■

Lemma 5. *A necessary condition for the symmetric matrix*

$$\begin{bmatrix} \boldsymbol{B} & \gamma \\ \gamma^{\mathsf{T}} & 1 \end{bmatrix}$$

to be positive semidefinite is $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{L}^{\mathsf{T}}$*, where* $\boldsymbol{L}$ *is (upper) triangular, while a sufficient condition is* $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{L}^{\mathsf{T}} + \gamma\gamma^{\mathsf{T}}$.

Proof: For necessity, note that if the complete matrix is positive semidefinite, then the upper left $n_q \times n_q$ submatix $\boldsymbol{B}$ must be as well. This implies the existence of a (possibly reduced rank) Choleski factorization $\boldsymbol{B}$ in the form $\boldsymbol{L}\boldsymbol{L}^{\mathsf{T}}$. For sufficiency, note that

$$\begin{bmatrix} \boldsymbol{B} & \gamma \\ \gamma^{\mathsf{T}} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}\boldsymbol{L}^{\mathsf{T}} + \gamma\gamma^{\mathsf{T}} & \gamma \\ \gamma^{\mathsf{T}} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L} & \gamma \\ \boldsymbol{0}^{\mathsf{T}} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{L}^{\mathsf{T}} & \boldsymbol{0} \\ \gamma^{\mathsf{T}} & 1 \end{bmatrix}. \qquad \blacksquare$$