CES PREFERENCES

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Abstract: The complete class of constant elasticity of substitution (CES) utility functions is additively separable with a power function aggregator over the sum of individual sub-utility functions. Each sub-utility function is positive-valued, increasing, and concave, but unrestricted in functional form. The added generality over the CES production function is due to ordinal utility, which does not imply constant returns to scale (CRS). Utility functions with identical level sets, demands, and welfare measures can have any elasticity of substitution. In both production and consumption, the elasticity of substitution contains no information about the degree of substitutability or complementarity among goods. Given CES, a method to nest and test for the additional property of CRS is identified.

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For a positive valued, twice continuously differentiable, increasing, and (quasi-)concave function, $u:\mathbb{R}_+^n \to \mathbb{R}_+, u \in \mathcal{C}^2,$ Hicks (1932, pp. 241-246) defines the elasticity of substitution between the two goods, q_i and q_j, by

$$\sigma_{ij} = \frac{\partial u/\partial q_i \cdot \partial u/\partial q_j}{u \cdot \partial^2 u/\partial q_i \partial q_j}, \; i \neq j. \tag{1}$$

When *u* is a production function with constant returns to scale (CRS), Arrow, Chenery, Minhas, and Solow (1961) and Uzawa (1962) show that the class of constant elasticity of substitution (CES) production functions is

$$u(q_1, \cdots, q_n) = \alpha \cdot \left(\beta_1 q_1^{(\sigma-1)/\sigma} + \cdots + \beta_n q_n^{(\sigma-1)/\sigma}\right)^{\sigma/(\sigma-1)}, \tag{2}$$

where $\alpha > 0,\;\; \beta_i > 0 \; \forall \; i,\;\; \sum_i \beta_i = 1,$ and $\sigma \neq 0,1.$ It is well-known (and easy to show) that $\sigma \to 1$ gives the Cobb-Douglas, $u = \alpha \cdot \prod_i q_i^{\beta_i}$, while $\sigma \to 0$ gives the Leontief, $u = \alpha \cdot \min\{\beta_1 q_1, \cdots, \beta_n q_n\}.$

This is the functional form of choice for consumer preferences in computable general equilibrium models (Robinson 1989). It also plays an important role in recent empirical applications of recursive models of consumption and asset pricing under uncertainty that do not satisfy the expected utility hypothesis (Epstein and Zin 1989, 1991; Epstein and Melino 1995).

However, in the theory of consumer choice, since utility functions are ordinal, there is no obvious reason to impose the CRS property. We would expect *a priori* that CRS plays an important part in the functional forms in equation (2) above. If so, then the set of all CES preferences is an open question, and this set may be considerably more general than the set of CRS + CES production functions.

The purpose of this letter is to show that the complete class of CES utility functions is

$$u(q_1,\cdots,q_n)=\begin{cases}\left[\left(\frac{\sigma-1}{\sigma}\right)\left(\beta\sum_{i=1}^{n}u_i(q_i)+\gamma\right)\right]^{\sigma/(\sigma-1)}, & \sigma\neq 1,\\ \exp\left\{\beta\sum_{i=1}^{n}u_i(q_i)+\gamma\right\}, & \sigma=1,\end{cases}\tag{3}$$

where the $u_i:\mathbb{R}_+\to\mathbb{R}_+$ are increasing, twice continuously differentiable, and concave (Gorman 1970, 1995), but otherwise unrestricted, and $\beta>0$. Thus, CES implies that preferences are additively separable, the CES property determines the functional form of the aggregator over individual sub-utility functions, and adding CRS to CES determines the functional form of the individual sub-utility functions.

Sufficiency is proven simply enough by differentiating (3) and substituting into (1), which gives $\sigma_{ij}=\sigma\ \forall\ i, j.$ To show necessity, let the elasticity of substitution be constant for the first two goods,

$$\frac{\partial u/\partial q_1\cdot\partial u/\partial q_2}{u\cdot\partial^2 u/\partial q_1\partial q_2}\equiv\sigma_{12}\ \ \forall\ \boldsymbol{q}\in\mathbb{R}_+^n.\tag{4}$$

Rearranging terms, this can be rewritten as

$$\frac{\partial}{\partial q_2}\ln\left(u\right)=\frac{\partial u/\partial q_2}{u}=\sigma_{12}\frac{\partial^2 u/\partial q_1\partial q_2}{\partial u/\partial q_1}=\sigma_{12}\frac{\partial}{\partial q_2}\ln\left(\partial u/\partial q_1\right).\tag{5}$$

Integrating the far left- and right-hand sides gives

$$\ln\left(u\right)=\sigma_{12}\ln\left(\partial u/\partial q_1\right)+c_1(q_1,q_3,\cdots,q_n).\tag{6}$$

By symmetry, we also can rewrite (4) as

$$\frac{\partial}{\partial q_1}\ln\left(u\right)=\frac{\partial u/\partial q_1}{u}=\sigma_{12}\frac{\partial^2 u/\partial q_1\partial q_2}{\partial u/\partial q_2}=\sigma_{12}\frac{\partial}{\partial q_1}\ln\left(\partial u/\partial q_2\right), \tag{7}$$

and integrate to obtain,

$$\ln\left(u\right)=\sigma_{12}\ln\left(\partial u/\partial q_2\right)+c_2(q_2,q_3,\cdots,q_n). \tag{8}$$

Taking the ratio of the exponent of (6) to the exponent of (8) then implies that

$$\left.\frac{\partial q_2}{\partial q_1}\right|_{u,q_3,\cdots,q_n}=-\frac{\partial u/\partial q_1}{\partial u/\partial q_2}=-\frac{\mathrm{e}^{c_1(q_1,q_3,\cdots,q_n)/\sigma_{12}}}{\mathrm{e}^{c_2(q_2,q_3,\cdots,q_n)/\sigma_{12}}}. \tag{9}$$

Separating the variables q_1 and q_2 and integrating implies that the utility function must be additively separable in these two goods,

$$u(q_1,\cdots,q_n)=f\left(u_1(q_1,q_3,\cdots,q_n)+u_2(q_2,q_3,\cdots,q_n),q_3,\cdots,q_n\right), \tag{10}$$

where $u_i(q_i,q_3,\cdots,q_n)=\int \mathrm{e}^{c_i(q_i,q_3,\cdots,q_n)/\sigma_{12}}\,dq_i$, $i=1,2$. Given (10), imposing a constant elasticity of substitution between q_1 and q_3, and so forth for all pairs of goods, then implies that utility is additively separable over all *n* goods,

$$u(q_1,\cdots,q_n)=f\left(u_1(q_1)+\cdots+u_n(q_n)\right). \tag{11}$$

The last step is to show that the CES property is then equivalent to choosing *f*. Differentiating (11) and reducing terms, CES for any pair of goods is equivalent to the second-order ordinary differential equation,

$$\frac{f'(x)^2}{f''(x)\cdot f(x)}=\sigma\ \forall\ x\in\mathbb{R}_+. \tag{12}$$

Similar to the above argument leading to (5), we can rewrite this as

$$\frac{d\ln f'(x)}{dx} = \frac{f''(x)}{f'(x)} = \frac{1}{\sigma}\frac{f'(x)}{f(x)} = \frac{1}{\sigma}\frac{d\ln f(x)}{dx}. \tag{13}$$

Direct integration implies

$$\ln f'(x) = \sigma^{-1}\ln f(x) + c. \tag{14}$$

Exponentiating and separating variables then gives

$$f'(x)\cdot f(x)^{-1/\sigma} = \beta, \tag{15}$$

where $\beta = \mathrm{e}^c > 0$. We isolate $\sigma = 1$, where $f'(x)/f(x) = d\ln f(x)/dx$, from the other cases, $\sigma \neq 1,$ where $f'(x)\cdot f(x)^{-1/\sigma} = \left(\frac{\sigma}{\sigma-1}\right)df(x)^{(\sigma-1)/\sigma}/dx$. Integrating (15) then gives

$$f(x) = \begin{cases} \left[\left(\frac{\sigma-1}{\sigma}\right)(\beta x + \gamma)\right]^{\sigma/(\sigma-1)}, & \sigma \neq 1, \\ \mathrm{e}^{\beta x + \gamma}, & \sigma = 1. \end{cases} \tag{16}$$

Substituting $\sum_i u_i(q_i)$ for x then gives equation (3) above. ■

Remarks

1. If we define $u_i(q_i) = \beta^{-1}\ln\tilde{u}_i(q_i)\ \forall\ i,$ we find that $\sigma = 1$ gives a multiplicative functional form (i.e., a generalized Cobb-Douglas) for consumer preferences, similar to the equivalent limiting case in production theory.
2. The constants β and γ do not play any significant role in defining the shape or structure of preferences, nor does the multiplicative constant $(\sigma-1)/\sigma$. One can normalize the sub-utility functions to absorb these constants, with no loss in generality.
3. Ordinal utility implies that indifference curves are invariant to raising to a power or exponentiating one utility function to obtain another. Hence, imposing the CES prop-

erty provides no information about the degree of substitutability or complementarity among goods. One can – with no loss in generality – construct a utility function with identical level sets, demands, and welfare measures for any real value of the elasticity of substitution.

4. In production theory, given CES, CRS can be nested in the production function by generalizing the functional form of the u_i to include the power functions of equation (2) as a special case (the null hypothesis). Conversely, modifying the exponent in the aggregator f changes the corresponding elasticity of substitution without altering the shape of the isoquants – though this does change their relative location. Hence, in the absence of CRS, the *elasticity of substitution* does not transmit any information about the degree of substitutability or complementarity among inputs, just like in the theory of consumption.

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